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Discovery of the wave nature of crystals

George Shpenkov

g.shpenkov@gmail.com

http://shpenkov.com/pdf/CrystalsNature.pdf

For many years of research, we have developed

A new general theory of physics

called

The Wave Model

(WM),

By now, it includes two specific theories (models):

(1) *Dynamic Model of elementary particles*

and

(2) *Shell-Nodal atomic model.*

Thanks to the WM, we have solved a number of problems accumulated in physics [1] **!**

The WM is beyond the Standard Model (SM).

New concepts, on which WM is based,
differ essentially from the relevant concepts underlying the SM:

The Wave Model

is based on

1. ***Dialectical philosophy and dialectical logic (dialectics)***

(in contrast to the formal logic accepted in modern physics)

and on

2. ***The axiom about the wave nature of all phenomena and objects in the Universe***

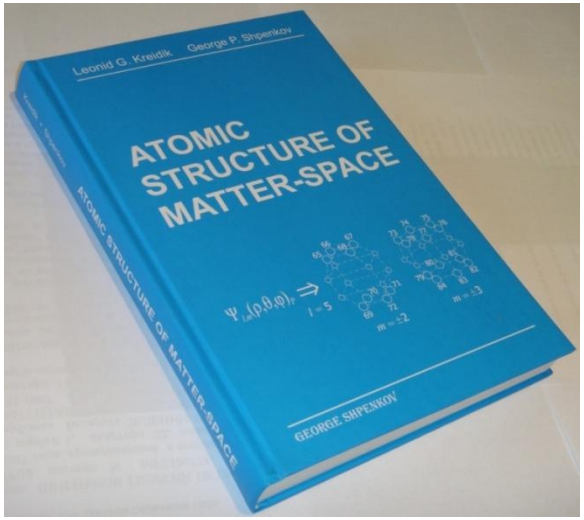
(It is the only axiom underlying the WM,
and no postulates are used in the WM!)

Whilst an unlimited number of abstract-mathematical
(fictional) postulates are in the basis of modern physics theories,
that is the main reason of all problems faced them.

The Wave Model

is described in detail in

a book



[L. G. Kreidik and G. P. Shpenkov, *Atomic Structure of Matter-Space*, Geo. S., Bydgoszcz, 2001, 584 p.]

http://shpenkov.com/atom.html

and

the lectures:

[G. P. Shpenkov, ***Dialectical View of the World. The Wave Model*** (Selected Lectures); Volumes 1-6]

http://shpenkov.com/pdf/Vol.1.Dialectics.pdf
... /Vol.2.DynamicModel-1.pdf
... /Vol.3.DynamicModel-2.pdf
... /Vol.4.PhysicalUnits.pdf
... /Vol.5.Shell-NodalAtomicStructure.pdf
... /Vol.6.TopicalIssues.pdf

From the WM it follows that

1. Atoms represent elementary molecules of hydrogen atoms !

(we consider as hydrogen atoms: proton (p), neutron (n) and protium (${}^{1}_{1}H$).

2. There are no superdense nuclei in the centers of atoms !

As a consequence of these key discoveries,
we came to a series of other discoveries.
In particular, we revealed that a carbon two-dimensional hexagonal lattice of

Graphene is anisotropic !

and has two-fold rotational symmetry, but not six-fold as is commonly believed.

I have **repeatedly reported** about these discoveries, in particular, in recent years at a number of international **conferences,** for example, in Paris (2016 [2]) and Berlin (2017 [3]).

Briefly about solutions

In accordance with the basic axiom of the WM, the structure of atoms as wave formations is described by well-developed methods of the physics of waves and, in particular, by the general ("classical") wave equation

$$\Delta\hat{\Psi} - \frac{1}{c^2}\frac{\partial^2\hat{\Psi}}{\partial t^2} = 0 \qquad (1)$$

This equation admits a particular solution of the form

$$\hat{\Psi}(\rho,\theta,\phi;t) = \hat{\psi}(\rho,\theta,\phi)e^{\pm i\omega t}, \qquad (2)$$

which describes the standing waves in a spherical space.

The spatial component in (2),

$$\hat{\psi}(\rho,\theta,\phi) = A\hat{R}_l(\rho)\Theta_{l,m}(\theta)\hat{\Phi}_m(\phi), \qquad (3)$$

is a particular solution of the time-independent form of Eq. (1) (Helmholtz equation):

$$\Delta\hat{\psi} + k^2\hat{\psi} = 0 \qquad (4)$$

Time-independent solution

$$\boxed{\hat{\psi}(\rho,\theta,\varphi) = A\hat{R}_l(\rho)\Theta_{l,m}(\theta)\hat{\Phi}_m(\varphi)} \quad (3)$$

determines the

shell-nodal structure of standing waves

(arrangement of nodes and antinodes)

in spherical space, and, as we revealed, the

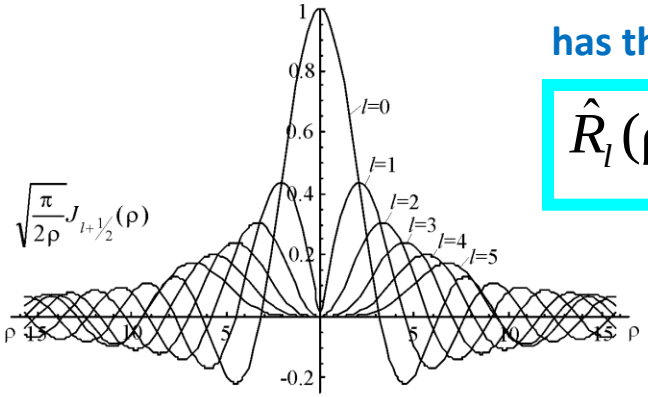
shell-nodal structure of "atoms" !

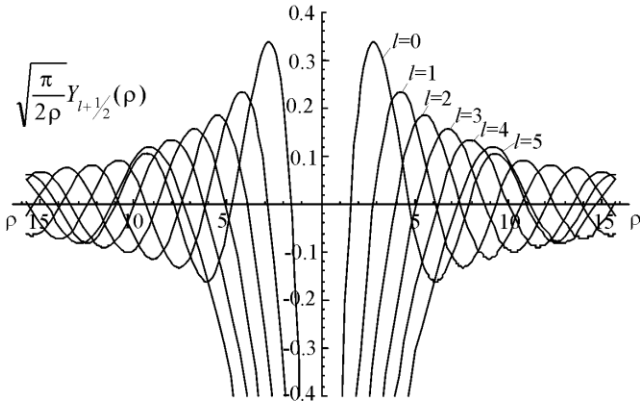
which turned out to be

Elementary Molecules of hydrogen atoms !

The explicit form of the $\hat{R}_l(\rho)$, $\Theta_{l,m}(\theta)$, $\hat{\Phi}_m(\varphi)$ components contained in (3) is as follows:

Radial component $\hat{R}_l(\rho)$ of the solution $\hat{\psi} = A\hat{R}_l(\rho)\Theta_{l,m}(\theta)\hat{\Phi}_m(\varphi)$





has the form:

$$\boxed{\hat{R}_l(\rho) = A\sqrt{\pi/2\rho}\,(J_{l+1/2}(\rho) \pm iY_{l+1/2}(\rho))} \qquad (5)$$

A is a constant factor;

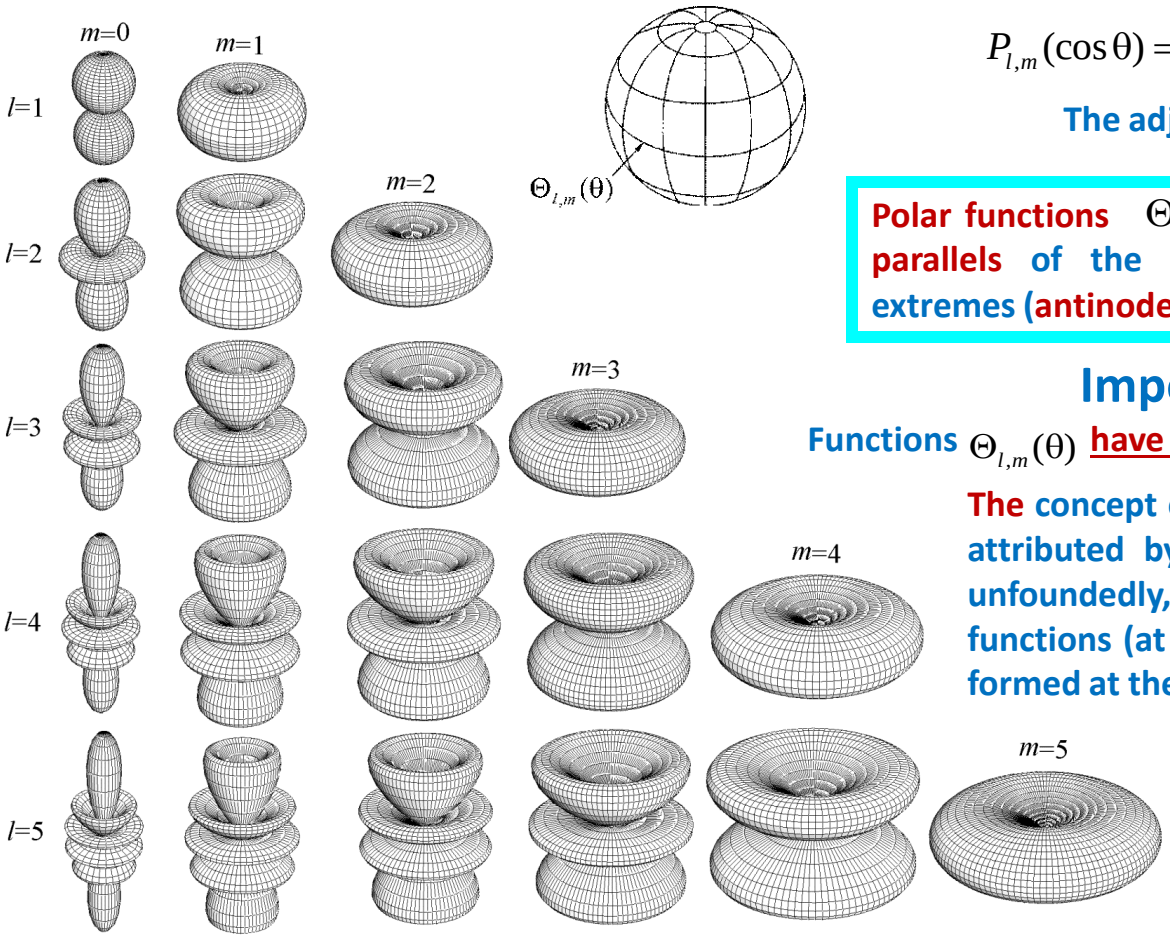
$l = 0,1,2,\ldots; \qquad m = 0, \pm 1, \pm 2, \ldots, \pm l$

Solutions of $\hat{R}_l(\rho)$ are roots $z_{\nu,q}$ (zeros and extremal values) of Bessel functions ***J*** and ***Y***, where $\nu = l + 1/2$ is the order of the functions , ***q*** is number of the zero or extremum.

Roots $z_{\nu,q}$ define the radii ***r*** of characteristic wave shells, potential and kinetic, on which are nodes and antinodes, respectively : $z_{\nu,q} = \rho_{\nu,q} = kr_{\nu,q}$, where $k = \omega_e / c$

Graphs of the polar functions

$$\left|\Theta_{l,m}(\theta)\right| = C_{l,m} \cdot P_{l,m}(\cos\theta) \qquad (6)$$



$$P_{l,m}(\cos\theta) = \frac{\sin^m\theta}{2^l\, l!}\frac{d^{l+m}}{d(\cos\theta)^{l+m}}\left(\cos^2\theta - 1\right)^l$$

The adjoined Legendre functions

Polar functions $\Theta_{l,m}(\theta)$ determine characteristic parallels of the location of zeros (nodes) and extremes (antinodes) on radial wave spherical shells.

Important notice!

Functions $\Theta_{l,m}(\theta)$ have nothing to do with "atomic orbitals".

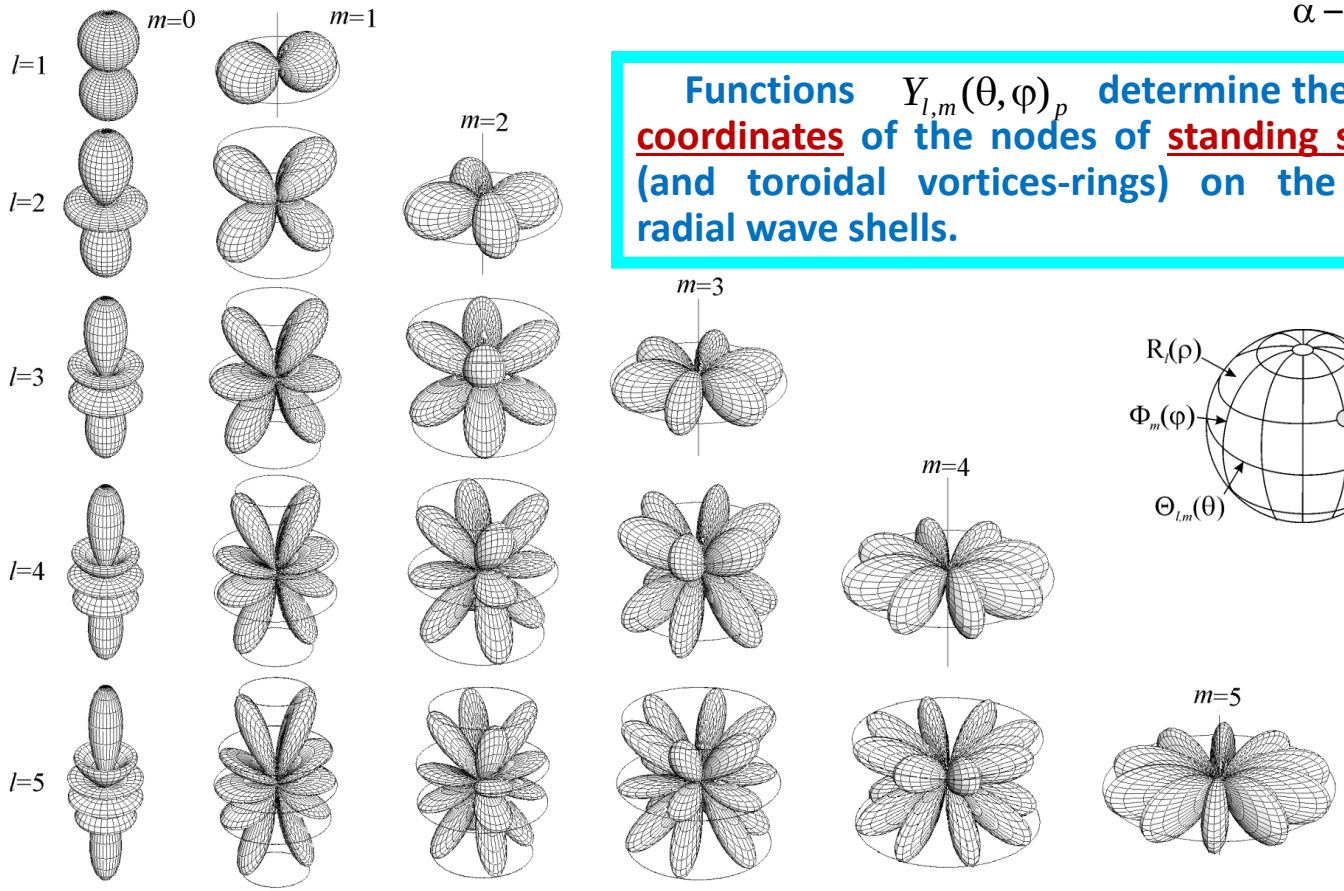
The concept of "atomic orbitals" was coined and attributed by founders of quantum mechanics unfoundedly, subjectively, to some of the functions (at l=1 and l=2), and to spatial figures formed at the rotation of their sections, and combinations thereof.

Graphs of the polar-azimuthal functions

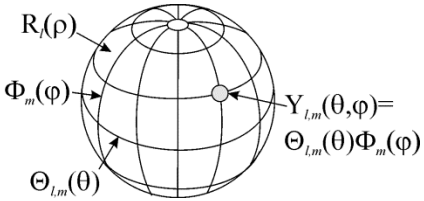
$$Y_{l,m}(\theta,\phi)_p = \left|\Theta_{l,m}(\theta)\,\mathrm{Cos}(m\varphi+\alpha)\right| \qquad (7)$$

(the potential component)

α – initial phase



Functions $Y_{l,m}(\theta,\varphi)_p$ determine the spatial angular coordinates of the nodes of standing spherical waves (and toroidal vortices-rings) on the corresponding radial wave shells.



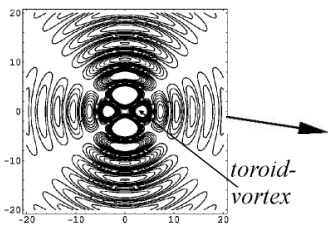
The potential component ψ_p of the particular solution (3) of the following explicit form

$$\psi_p = A\sqrt{\frac{\pi}{2\rho}}\, J_{l+1/2}(\rho) P_{l,m}(\cos\theta)\,\mathrm{Cos}(m\varphi+\alpha) \qquad (8)$$

determines disposition of nodes and toroidal vortices of standing waves in spherical space

The schematic representation of the solution (8) shown here was unknown earlier. We did this for the first time in physics.

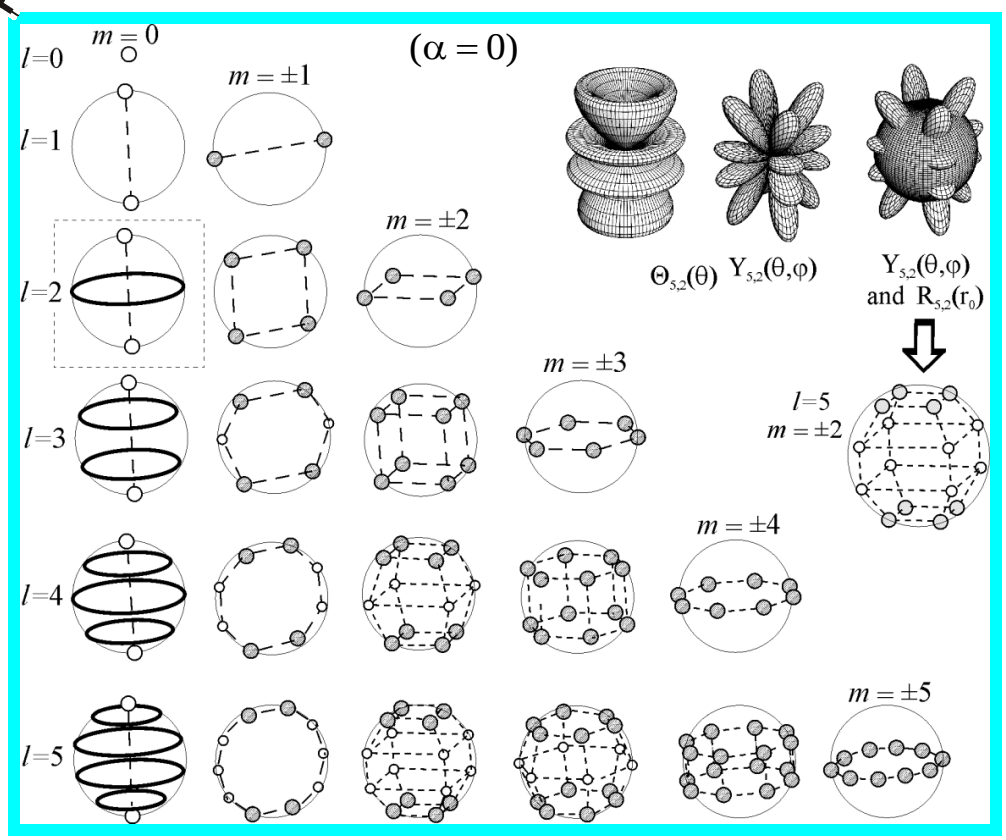
The solution for $l = 2,\ m = 0$ (section $x = 0$)



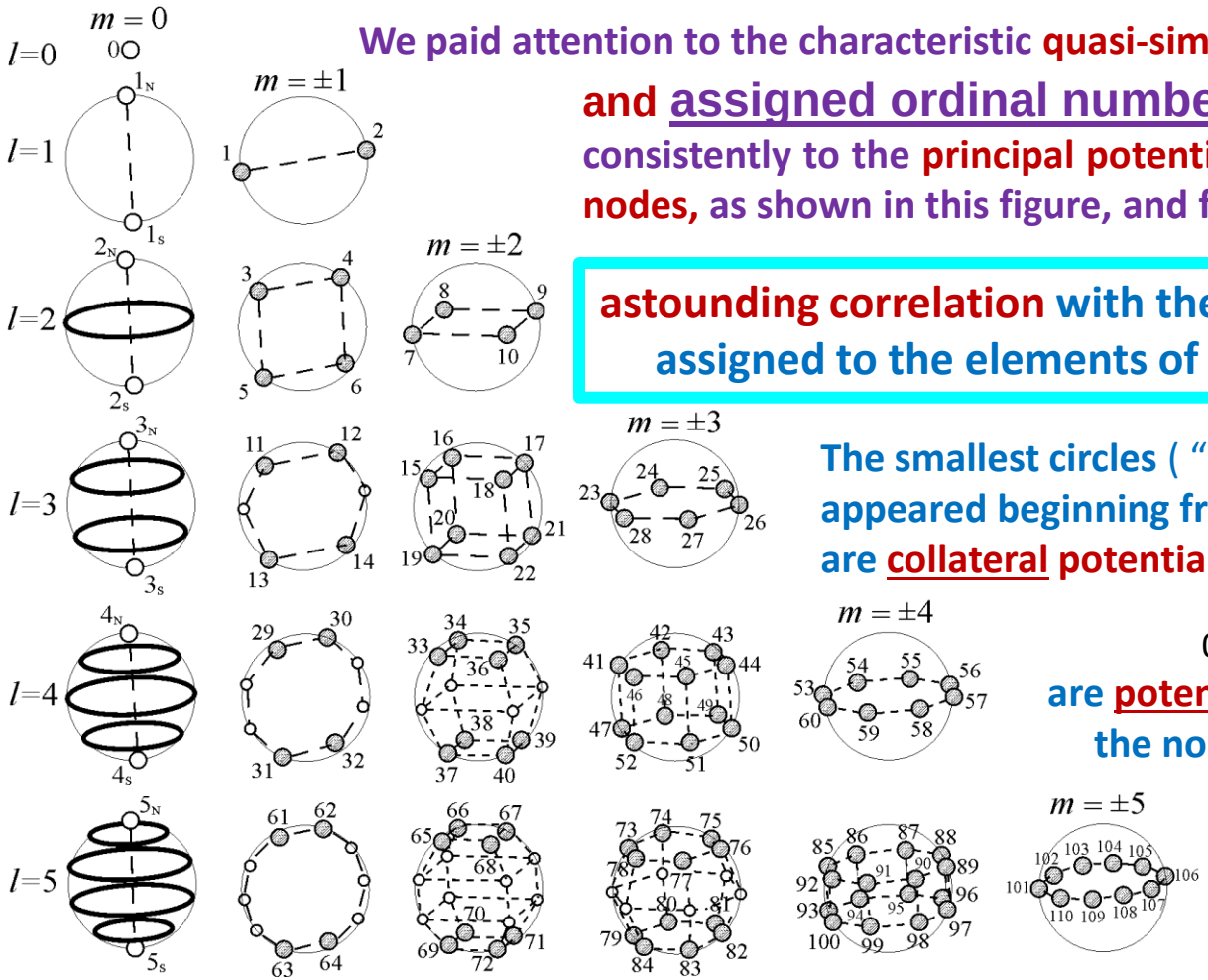
This presentation was key and led us in the course of comprehensive long-term studies (the results of which were first published in 1996 [5]), to the key discovery that this solution determines the shell-nodal structure of the "atoms", which, being the wave formations, are in fact

Elementary Molecules of hydrogen atoms.

Shells ($l = 0,1,2,\ldots$) and subshells ($m = 0,\pm1,\pm2,\ldots,\pm l$)



How we have come to this discovery?



We paid attention to the characteristic quasi-similarity, periodic trends…

and assigned ordinal numbers 1, 2, 3,. .., 110, … !

consistently to the principal potential polar-azimuthal wave nodes, as shown in this figure, and found their

astounding correlation with the atomic numbers Z assigned to the elements of the Periodic Table !

The smallest circles (“o “) (empty, unnumbered) appeared beginning from $l = 3$, $m = \pm 1$, are collateral potential polar-azimuthal nodes.

0, 1_N, 1_S, …, 5_N, 5_S are potential-kinetic polar nodes – the nodes of rest and motion simultaneously.

Analyzing characteristic features of the solutions (8), presented in this way (with numbered nodes), we made sure, ultimately, that they really give us information about the shell-nodal (molecule-like) structure of the “atoms” !

Solution (8) determines

the nodal structure of standing waves, and, indeed, as we revealed,

the nodal structure of all fully completed wave shells of the "atoms",

elementary molecules of hydrogen atoms,

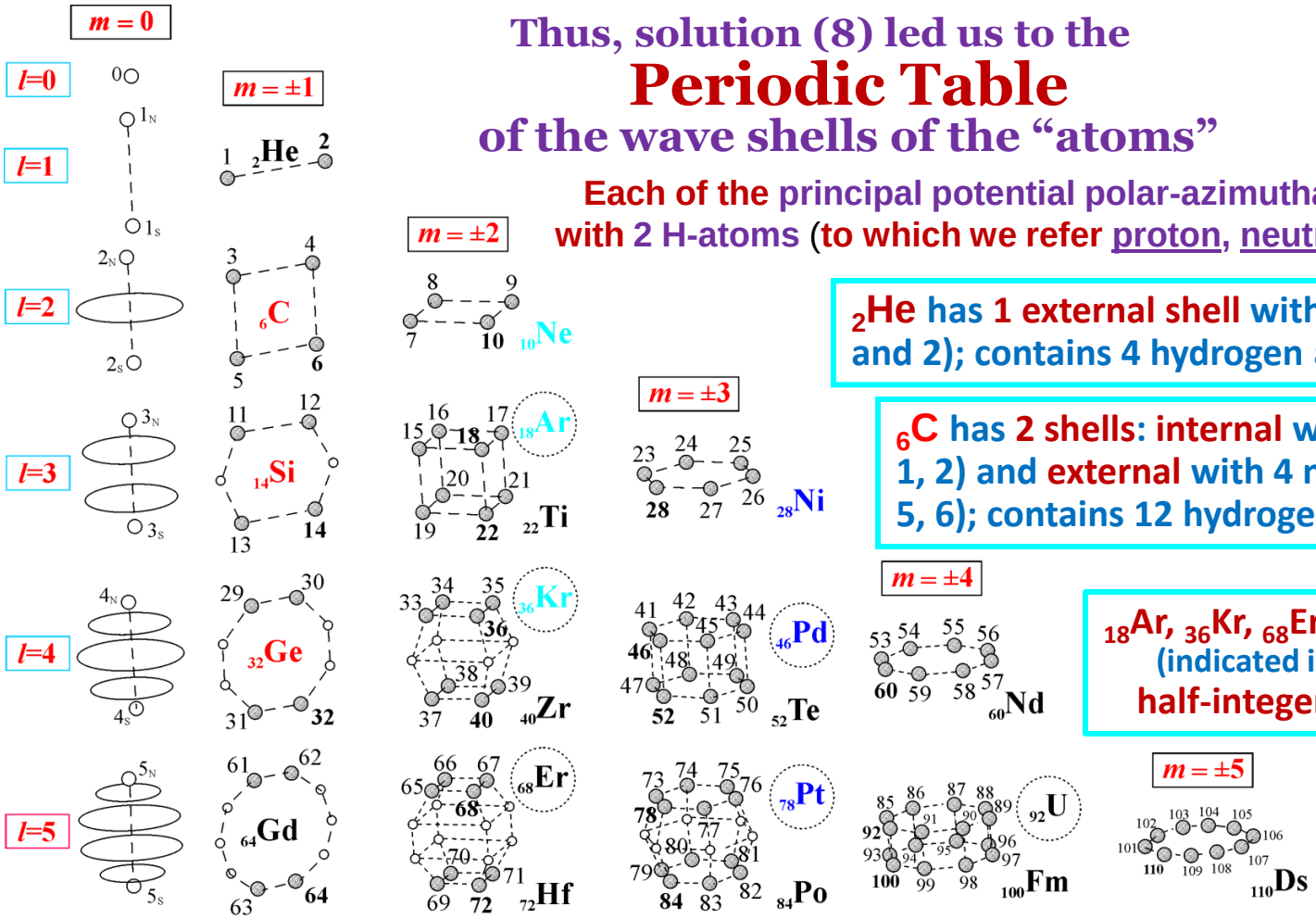
as shown here.

Thus, solution (8) led us to the

Periodic Table

of the wave shells of the "atoms"

Each of the principal potential polar-azimuthal nodes is filled with 2 H-atoms (to which we refer proton, neutron, and protium).



$_2$He has 1 external shell with 2 nodes (№ 1 and 2); contains 4 hydrogen atoms (2x2).

$_6$C has 2 shells: internal with 2 nodes (№ 1, 2) and external with 4 nodes (№ 3, 4, 5, 6); contains 12 hydrogen atoms (6x2).

$_{18}$Ar, $_{36}$Kr, $_{68}$Er, $_{46}$Pd, $_{78}$Pt, $_{92}$U (indicated in circles) have half-integer outer shells.

Noninteger solutions

$$l = m = s/2, \quad s \in \mathbf{N},$$

correspond to intermediate states. They define uncompleted external shells and subshells: half-completed (shown above) and partially completed, such as the shells of

$_3$Li, $_4$Be, $_5$B or $_7$N, $_8$O, $_9$F, etc.

$$\hat{\psi} = A\sqrt{\frac{\pi}{2\rho}}\left(J_{\frac{s}{2}+\frac{1}{2}}(\rho) \pm iY_{\frac{s}{2}+\frac{1}{2}}(\rho)\right)\sin^{\frac{s}{2}}\theta(\cos\frac{s}{2}\varphi \pm i\sin\frac{s}{2}\varphi) \qquad (9)$$

Zeros and extremes of the noninteger solutions (nodes and antinodes) are in the

equatorial plane (z=0)

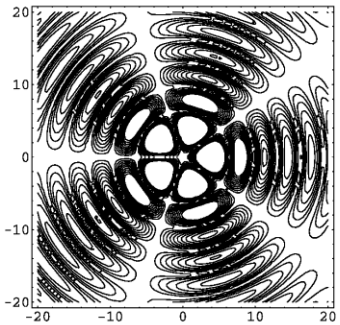
and have

any-fold symmetry,

including <u>forbidden by mathematical laws of crystallography</u>. Here are two examples.

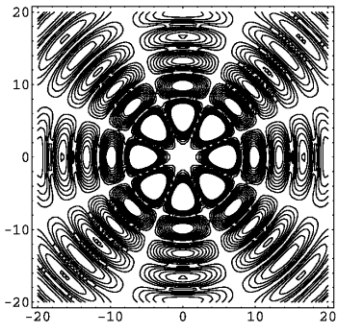
Contour plots of the sections for the potential component of the solutions, determined by the function:

$$\boxed{\frac{J_{s/2+1/2}(\rho)}{\sqrt{\rho}}\sin^{s/2}\theta\cos(\tfrac{s}{2}\varphi)} \qquad (10)$$



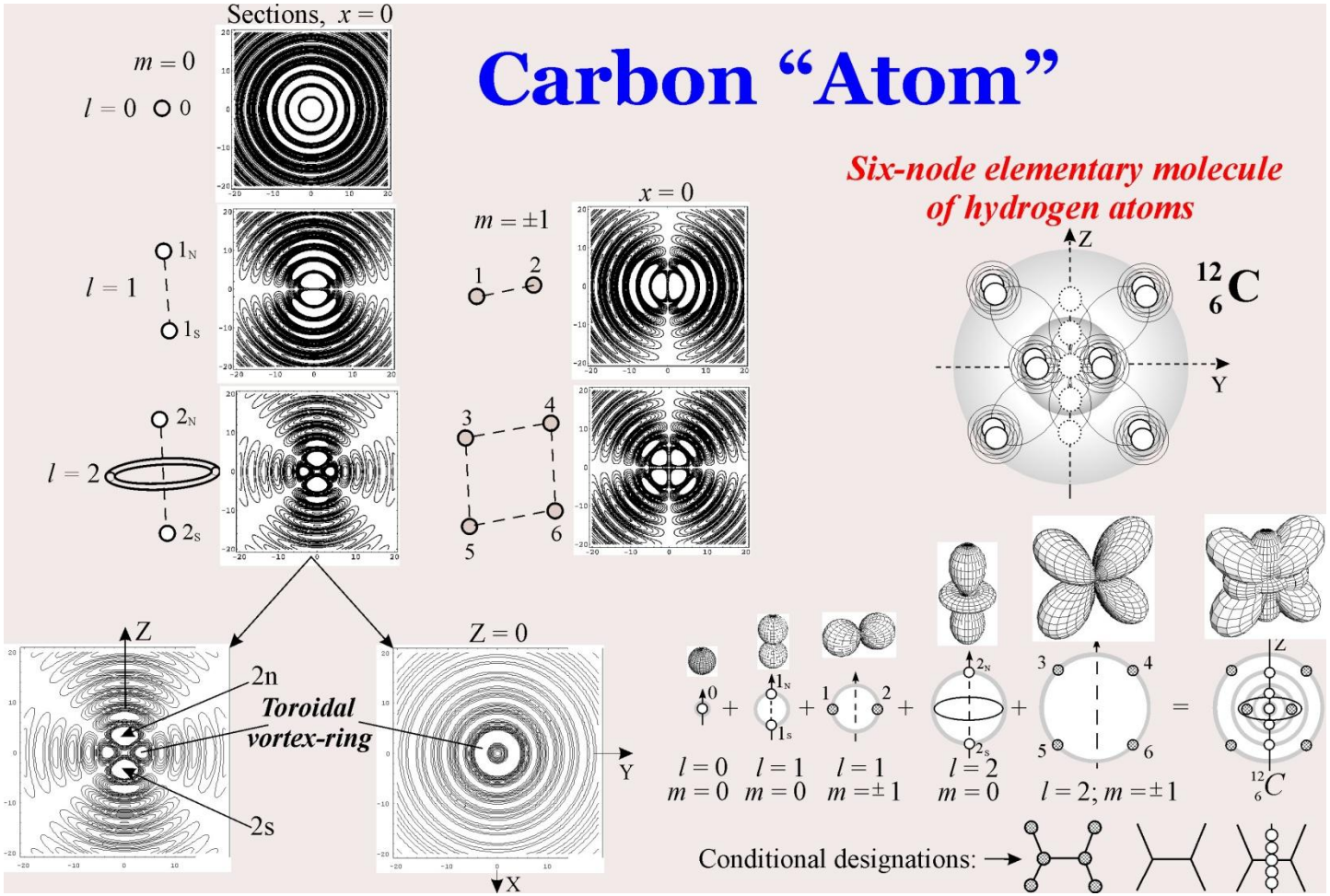
S=5

S=8

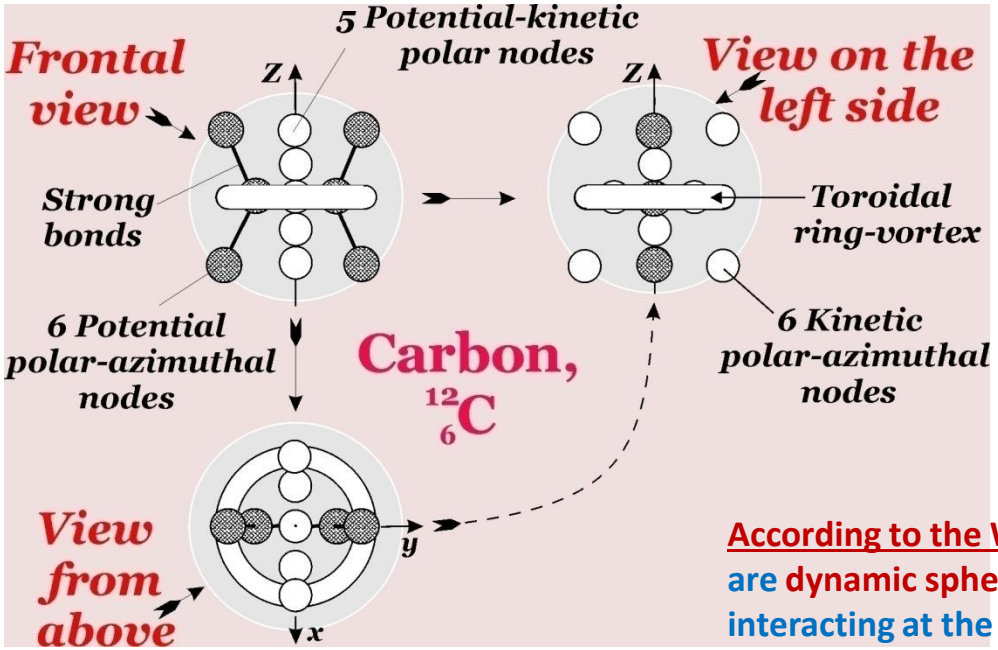


Equatorial distribution of the nodes of the five- and eight-fold symmetries

Shell-nodal structure of $^{12}_{6}C$ – six-node elementary molecule of H-atoms



Six-node elementary molecule of hydrogen atoms (carbon "atom"), ${}^{12}_{6}C$



Potential and kinetic polar-azimuthal nodes dislocated relative to each other in the radial direction, and are in planes differing in phase by $\varphi=\pi/2$

Fundamental frequency and the fundamental wave radius of the atomic and subatomic levels:

$$\omega_e = 1.869162559\times10^{18}\ s^{-1}$$

$$\lambda\!\!\!^{-}_e = \frac{c}{\omega_e} = 1.603886492\times10^{-8}\ cm$$

According to the WM, all elementary particles and atoms are dynamic spherical formations pulsating and interacting at the frequency ω_e (discovered in the WM, along with ω_g and E_B).

Binding energy of the nodes in ${}^{12}_{6}C^{-4}$, calculated by the formula is $E_{C,ion} = 92.349...\text{MeV}$

$$E_B = \omega_e^2 \frac{m_1 m_2}{8\pi\varepsilon_0 r}$$

$\omega_e m_2 = q_1$ and $\omega_e m_1 = q_1$ are exchange charges of interacting nucleons; $\varepsilon_0 = 1 g\cdot cm^{-3}$ is the absolute unit density.

Fundamental frequency and the fundamental wave radius of the gravitational level:

$$\omega_g = 9.158082264\times10^{-4}\ s^{-1}$$

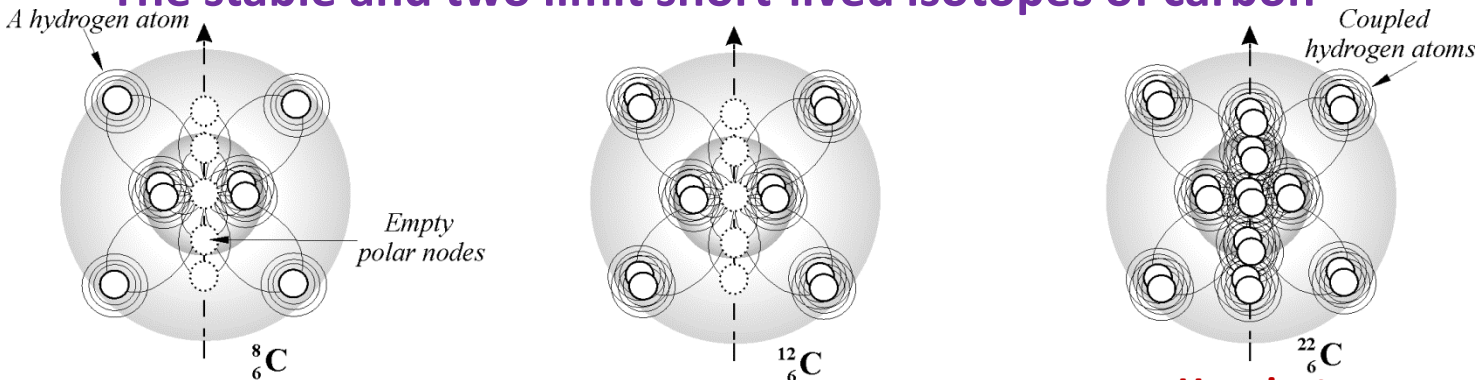
$$\lambda\!\!\!^{-}_g = \frac{c}{\omega_g} = 327.4\times10^{6}\ km$$

Discovery of the shell-nodal structure of the atoms allowed us to reveal

The nature of origin and the structure of all "atomic" isotopes

Here is an example:

The stable and two limit short-lived isotopes of carbon*



Lightest
(The least stable isotope, a half-life of 2.0×10^{-21} s)

Stable

Heaviest
(A half-life of 6.2×10^{-3} s)

*[G. P. Shpenkov, *Physics and Chemistry of Carbon in the Light of Shell-Nodal Atomic Model*, Chapter 12 in "*Quantum Frontiers of Atoms and Molecules*", edited by Putz M. V., NOVA SCIENCE PUBLISHERS, NY, 277-323, 2011]

It is another in a series of the discoveries following from the fundamental discovery (the shell-nodal structure of the "atoms") and, thereby, directly confirming reality of the latter.

“Atomic” isotopes

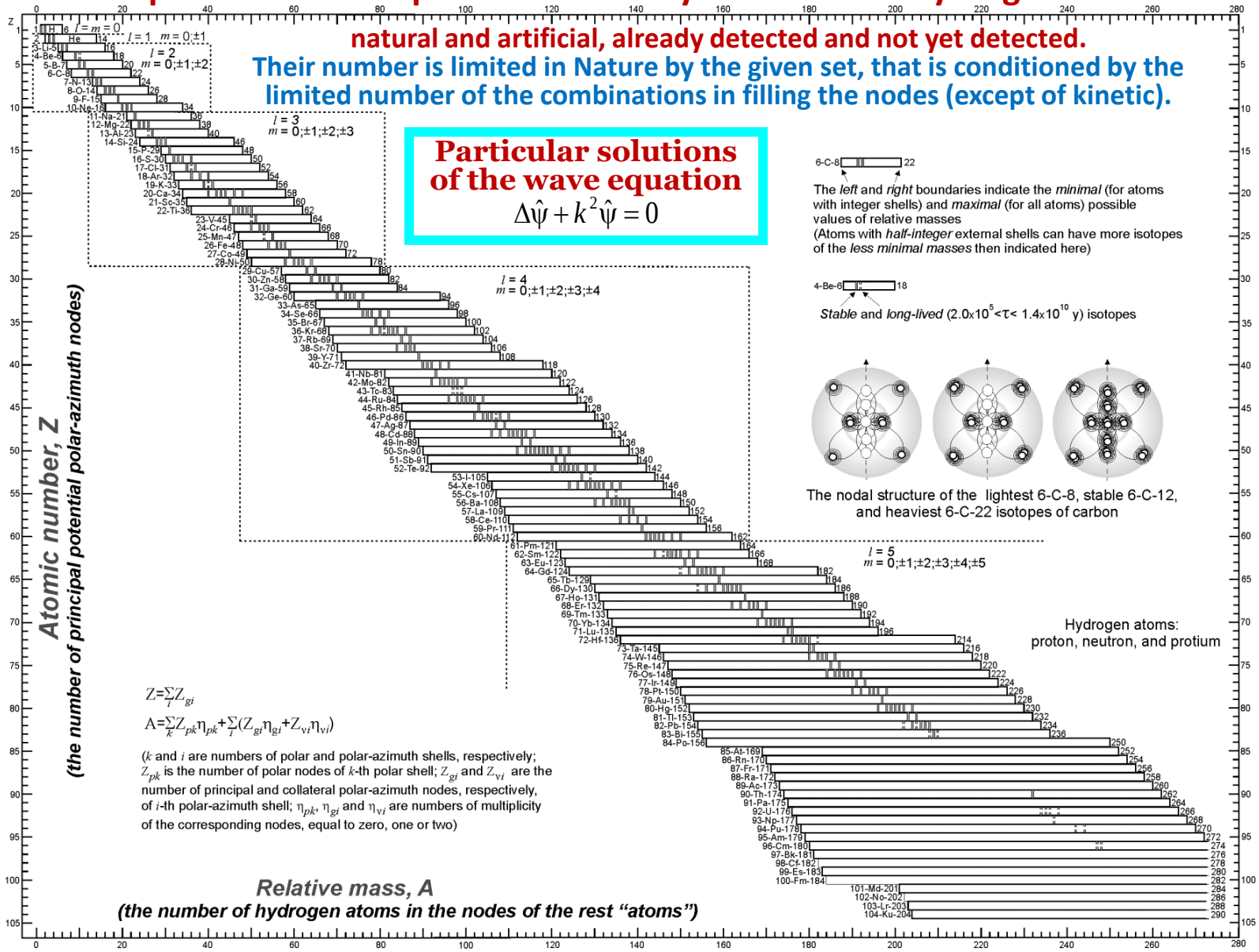
Complete set of isotopes of elementary molecules of hydrogen atoms

natural and artificial, already detected and not yet detected.

Their number is limited in Nature by the given set, that is conditioned by the limited number of the combinations in filling the nodes (except of kinetic).

Particular solutions of the wave equation

$$\Delta\hat{\psi} + k^2\hat{\psi} = 0$$



$$Z = \sum_i Z_{gi}$$

$$A = \sum_k Z_{pk}\eta_{pk} + \sum_i (Z_{gi}\eta_{gi} + Z_{vi}\eta_{vi})$$

(k and i are numbers of polar and polar-azimuth shells, respectively; Z_{pk} is the number of polar nodes of k-th polar shell; Z_{gi} and Z_{vi} are the number of principal and collateral polar-azimuth nodes, respectively, of i-th polar-azimuth shell; η_{pk}, η_{gi} and η_{vi} are numbers of multiplicity of the corresponding nodes, equal to zero, one or two)

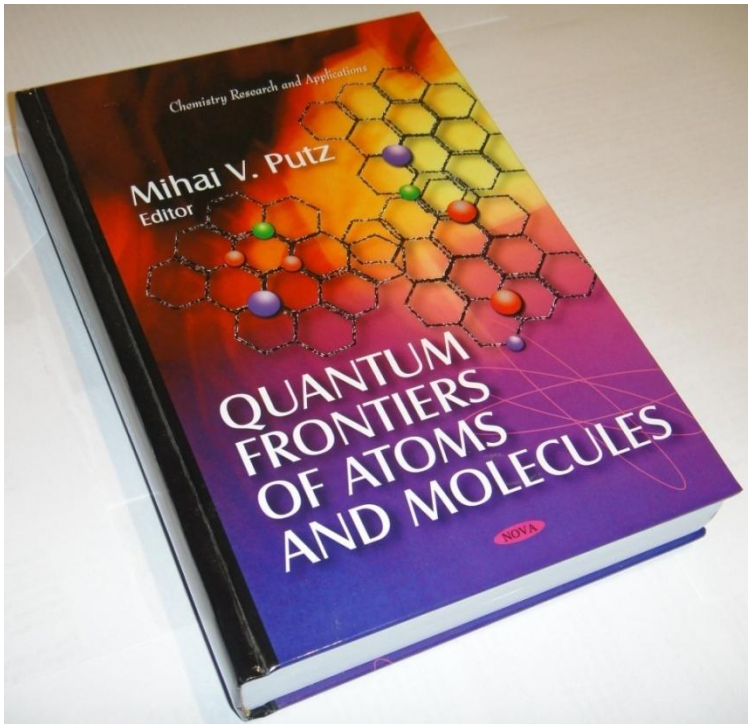
The *left* and *right* boundaries indicate the *minimal* (for atoms with integer shells) and *maximal* (for all atoms) possible values of relative masses
(Atoms with *half-integer* external shells can have more isotopes of the *less minimal masses* then indicated here)

Stable and *long-lived* ($2.0\times10^5 < \tau < 1.4\times10^{10}$ y) isotopes

The nodal structure of the lightest 6-C-8, stable 6-C-12, and heaviest 6-C-22 isotopes of carbon

Hydrogen atoms:
proton, neutron, and protium

[G. P. Shpenkov, *Physics and Chemistry of Carbon in the Light of Shell-Nodal Atomic Model*, Chapter 12 in "*Quantum Frontiers of Atoms and Molecules*", edited by Putz M. V., NOVA SCIENCE PUBLISHERS, NY, 277-323, 2011]

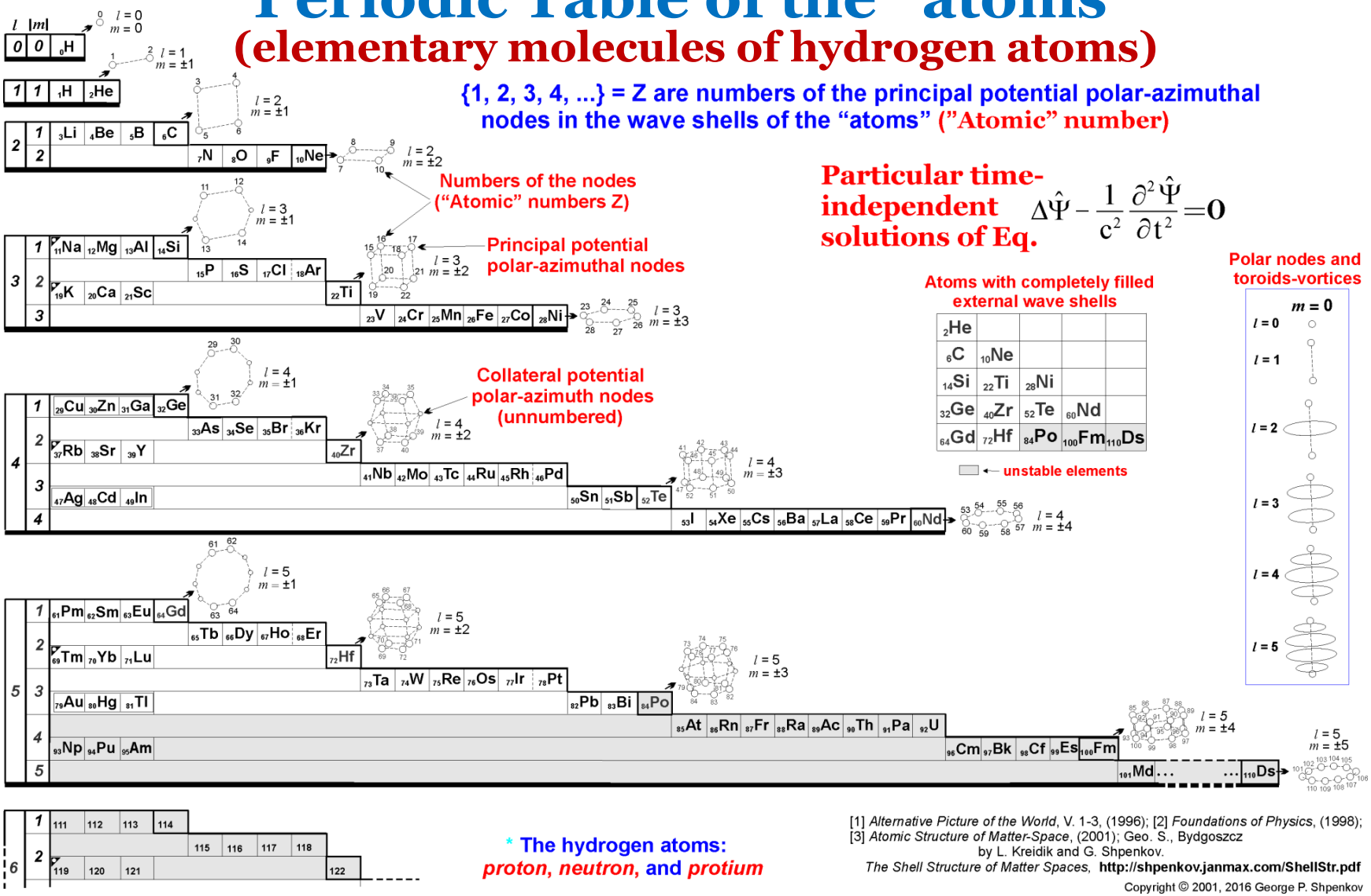


A colour variant of the Table of Isotopes of 2001
is available online at http://shpenkov.com/pdf/isotopes.pdf

Periodic Table of the "atoms"

(elementary molecules of hydrogen atoms)

{1, 2, 3, 4, ...} = Z are numbers of the principal potential polar-azimuthal nodes in the wave shells of the "atoms" ("Atomic" number)



Particular time-independent solutions of Eq.

$$\Delta\hat{\Psi} - \frac{1}{c^2}\frac{\partial^2\hat{\Psi}}{\partial t^2} = 0$$

Atoms with completely filled external wave shells

2He				
6C	10Ne			
14Si	22Ti	28Ni		
32Ge	40Zr	52Te	60Nd	
64Gd	72Hf	84Po	100Fm	110Ds

unstable elements

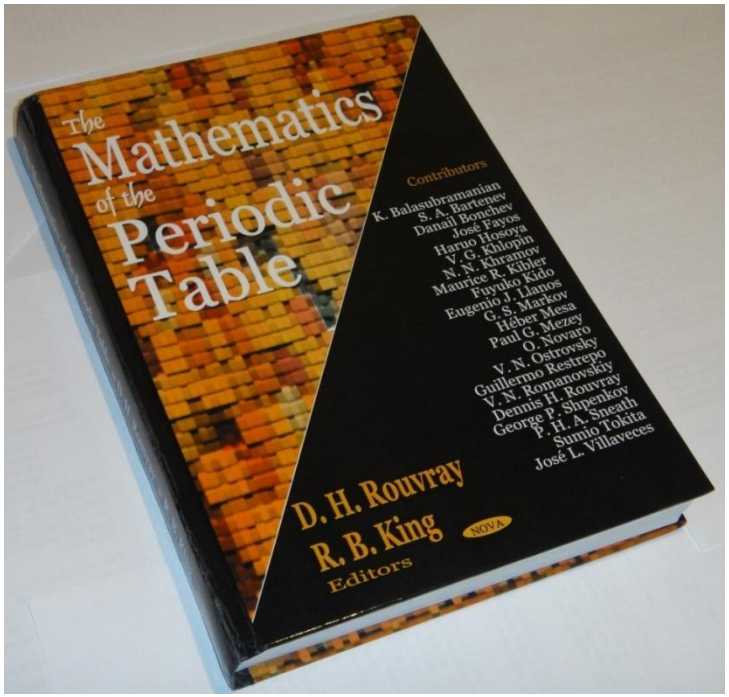
Polar nodes and toroids-vortices

* **The hydrogen atoms:**
proton, neutron,* and *protium

[1] *Alternative Picture of the World*, V. 1-3, (1996); [2] *Foundations of Physics*, (1998);
[3] *Atomic Structure of Matter-Space*, (2001); Geo. S., Bydgoszcz
by L. Kreidik and G. Shpenkov.
The Shell Structure of Matter Spaces, **http://shpenkov.janmax.com/ShellStr.pdf**

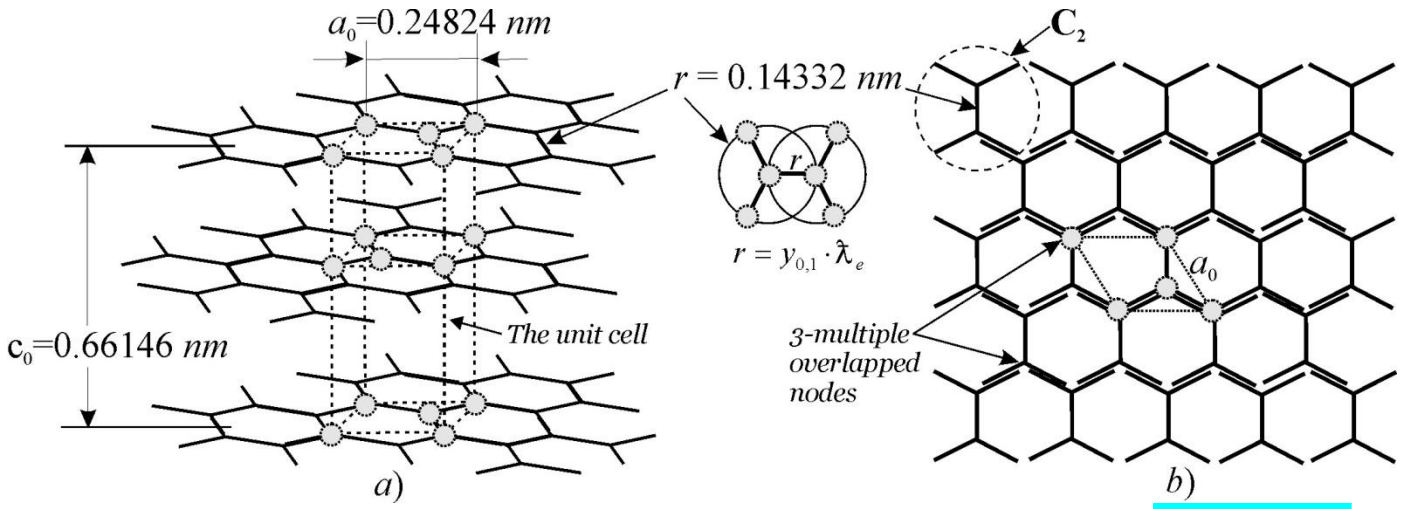
Thanks to the discovery of molecule-like structure of atoms, the true primary cause of the observed similarity in physical and chemical properties of elements (first generalized and formulated by D. I. Mendeleev in his Periodic Law) was also revealed.

[G. P. Shpenkov, *An Elucidation of the Nature of the Periodic Law*, Chapter 7 in "*The Mathematics of the Periodic Table*", edited by Rouvray D. H. and King R. B., NOVA SCIENCE PUBLISHERS, NY, 119-160, 2006]



A colour variant of the Periodic Table of 2001
is available online at http://shpenkov.com/pdf/placard.pdf

The structure of graphite (a), and its only atomic layer – graphene (b)



According to WM, the length of bonds in graphite is determined by the product $\boxed{r = y_{0,1} \cdot \lambda\!\!\!^{-}_e}$, where $y_{0,1}$ is the root of Bessel functions [6] ($y_{0,1}$= 0.89357697) and $\lambda\!\!\!^{-}_e = \omega_e / c = 1.603886538 \cdot 10^{-8}\ cm$

$$n = (1\tfrac{1}{6} \times 2 + 1\tfrac{2}{3}) = 4\ nodes\ per\ unit\ cell,\ V = n \cdot 12.0107 \cdot m_u / \rho = 35.29953318 \cdot 10^{-24}\ cm^3,$$
$$m_u = 1.660539040 \cdot 10^{-24} g,\ \rho = 2.26\ g \cdot cm^{-3},\ a_0 = \sqrt{3} r,\ c_0 = V \cdot 2 / a_0^2 \sqrt{3} = 6.614634572 \cdot 10^{-8}\ cm$$

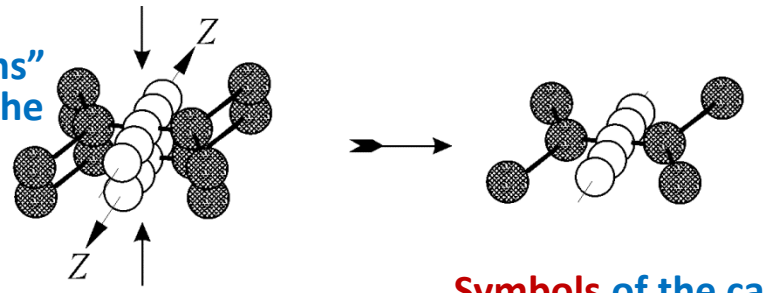
Lattice constants a_0 and c_0, calculated as shown here, are close to the lattice constants of graphite (at 300 K) known from the literature. The calculation data correspond to graphite consisting of carbon dimers C_2 having the shell-nodal structure.

Thus, we have come to the conclusion that the "building blocks" of graphite and, hence, graphene are carbon dimers C_2.

Schematic representation of the formation of the molecule C_2

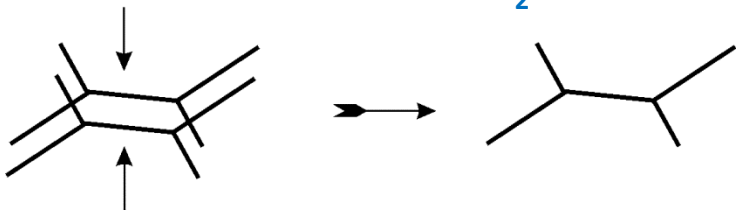
C_2 is formed by overlapping, "merging", all approaching nodes (and toroidal rings-vortices, not shown here) of two 6-nodal elementary molecules of hydrogen atoms (two carbon "atoms") into a single whole.

Like magnets, two carbon "atoms" are attracted to each other by the opposite directed fields of their internal internodal bonds



$$\mathbb{C}+\mathbb{C} \longrightarrow \mathbb{C}_2$$

Symbols of the carbon "atom" C and the molecule C_2 are the same



In [H. C. Shih, et al., *Diamond and Related Materials*, 2, 531 (1993)],

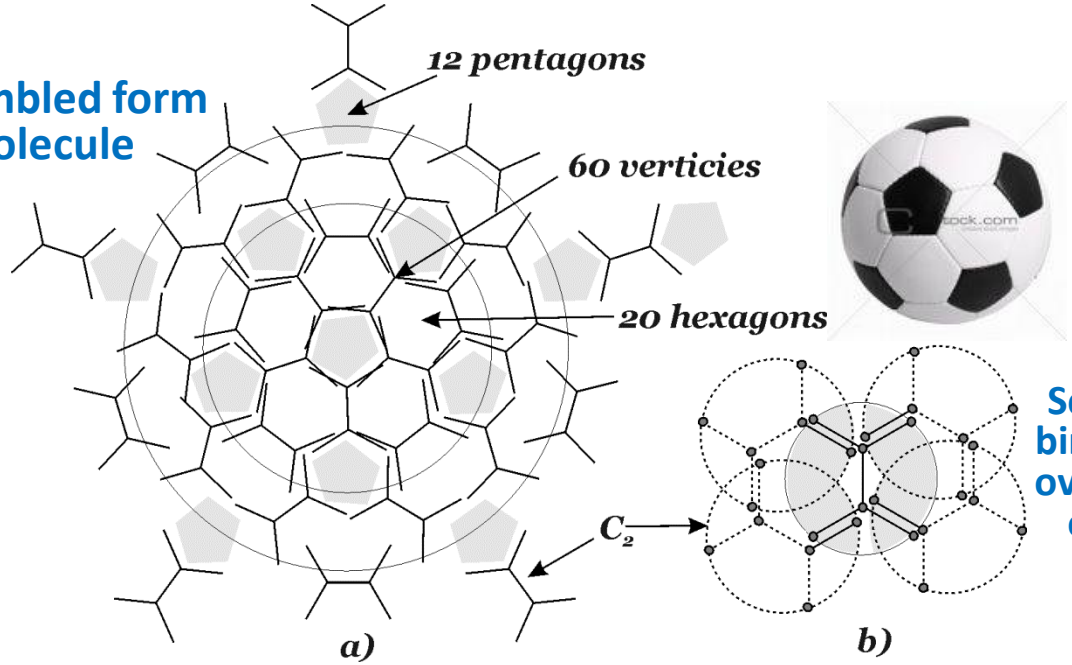
"...the C_2 radical was considered to be responsible for the formation of graphite"

Buckminsterfullerene (C_{60}),

in accordance with the WM,

is formed from 30 carbon dimers, C_2 :

The disassembled form of the molecule



Scheme of binding and overlapping of nodes

Thus, according to WM, the formula of the molecule is $(C_2)_{30}$

"Carbon Dimer (C_2) is in fact the major observable product of C_{60} fragmentation. Being a very effective growth species, it can rapidly incorporate into the diamond lattice leading to high-film growth rates"*.

*[D. M. Gruen, et al., *Turning Soot Into Diamonds With Microwaves*, Proceedings of the 29th Microwave Power Symposium, Chicago, Illinois, July 25-27, 1994]

A diagram showing

Formation of chemical ("covalent") C–C bonds

in hydrocarbon compounds

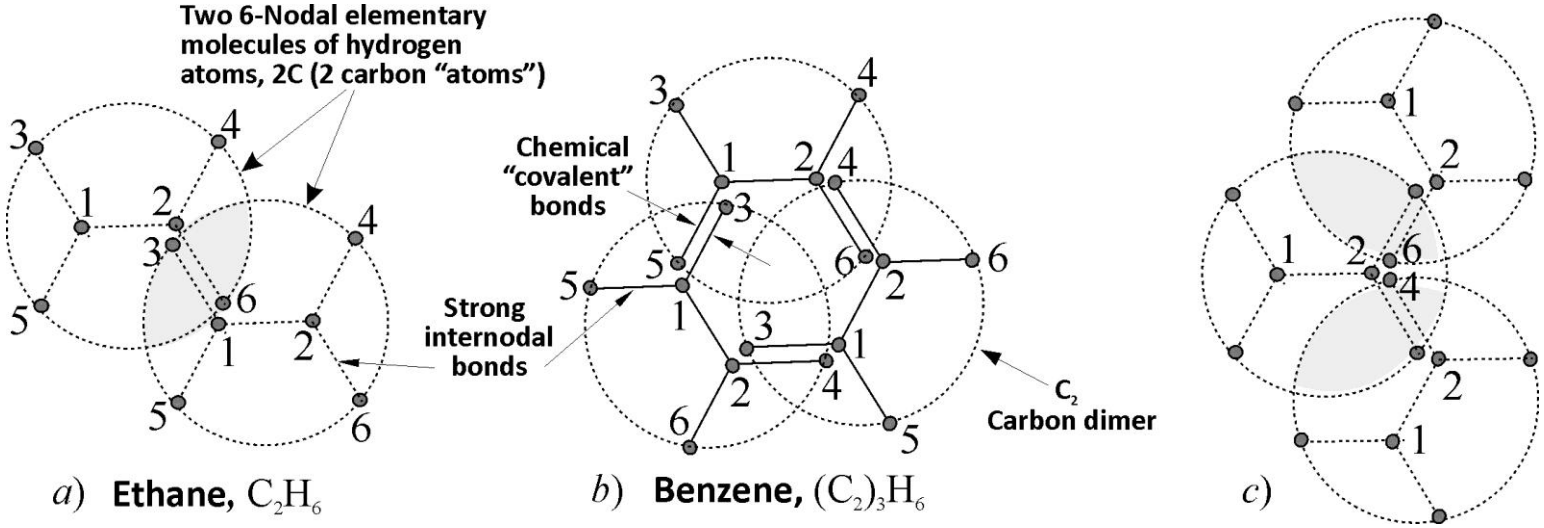
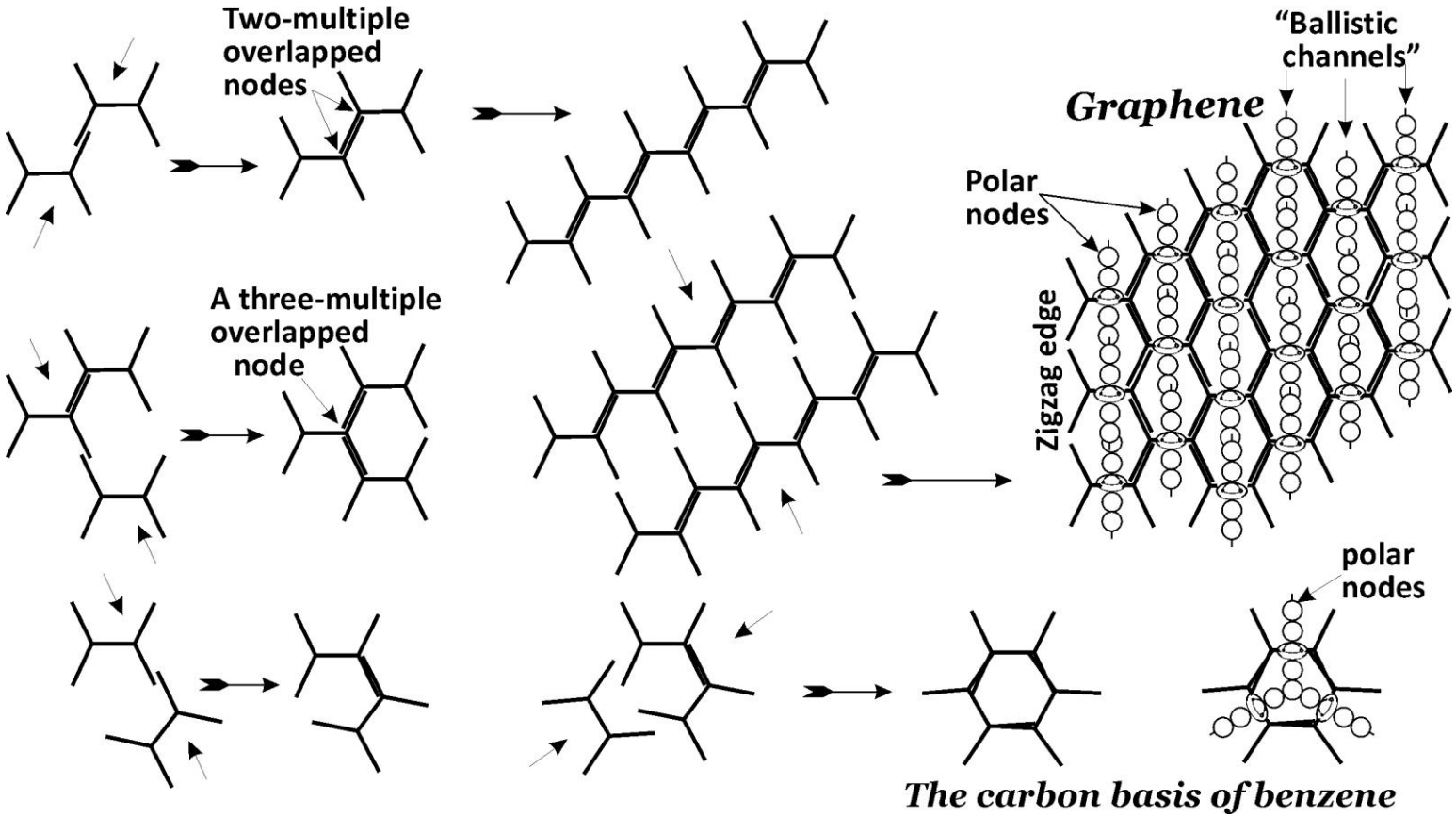


Fig. Two-multiple (a, b) and three-multiple (c) overlapping of external and internal polar-azimuthal nodes, belonging to the joined carbon nucleon molecules (single, C, or dimmers, C_2).

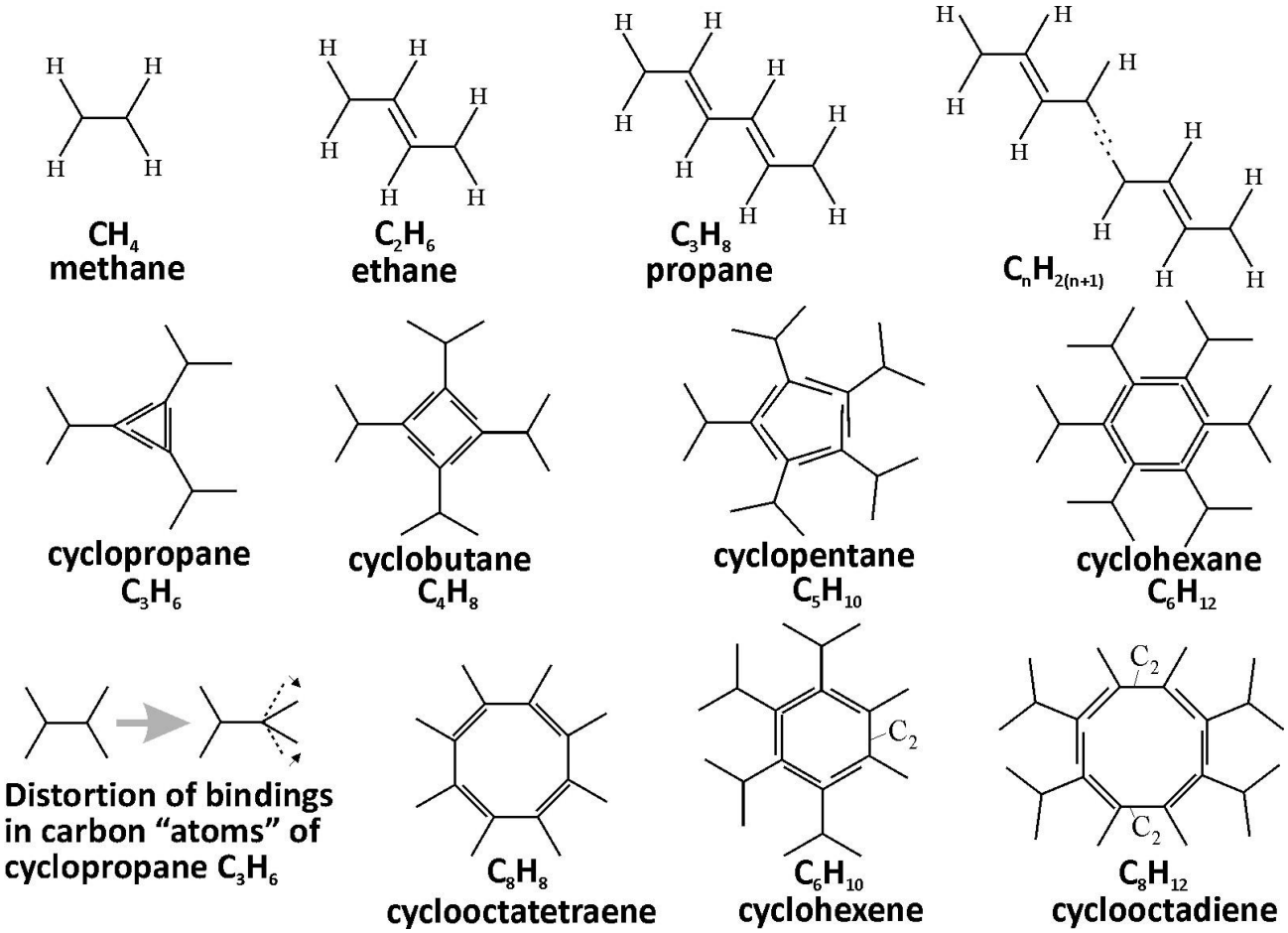
Chemical "covalent" bonds are realized along the lines of strong internodal bindings (existing between external and internal nodes) each of the joined "atoms" (a) or dimers (b). Electrons play the secondary role, they define only the strength of the bonds.

Schematic representation of self-bindings (assembling) of two-dimensional carbon compounds



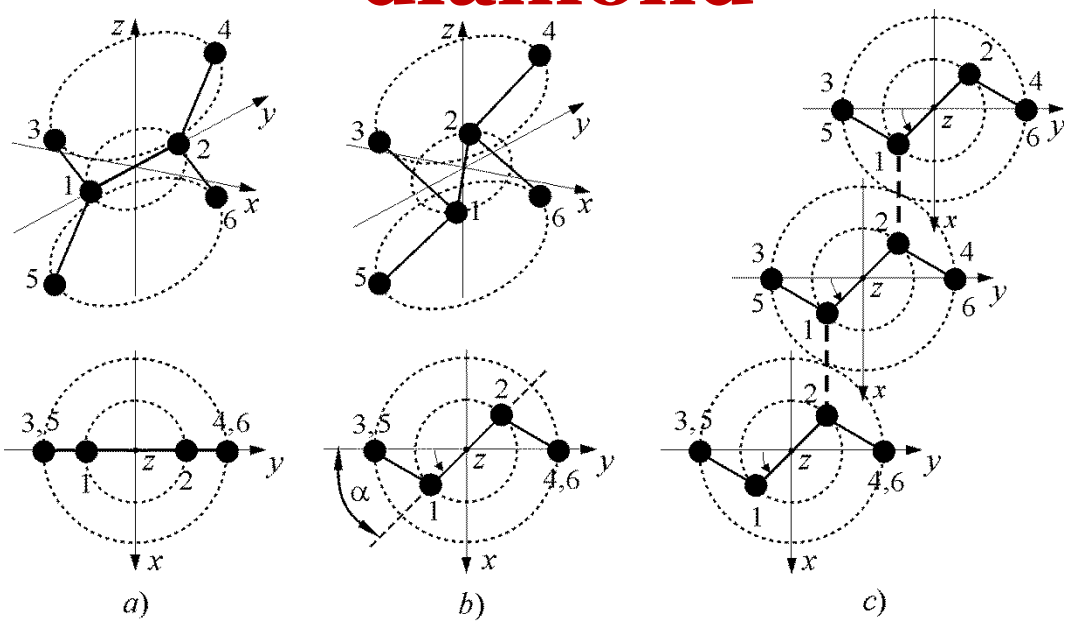
The structure of C-C bonds

in typical hydrocarbon compounds*



*[G. P. Shpenkov, The Role of Electrons in Chemical Bonds Formations (In the Light of Shell-Nodal Atomic Model), Molecular Physics Reports 41, 89-103, (2005)]

Formation of internodal bonds in diamond



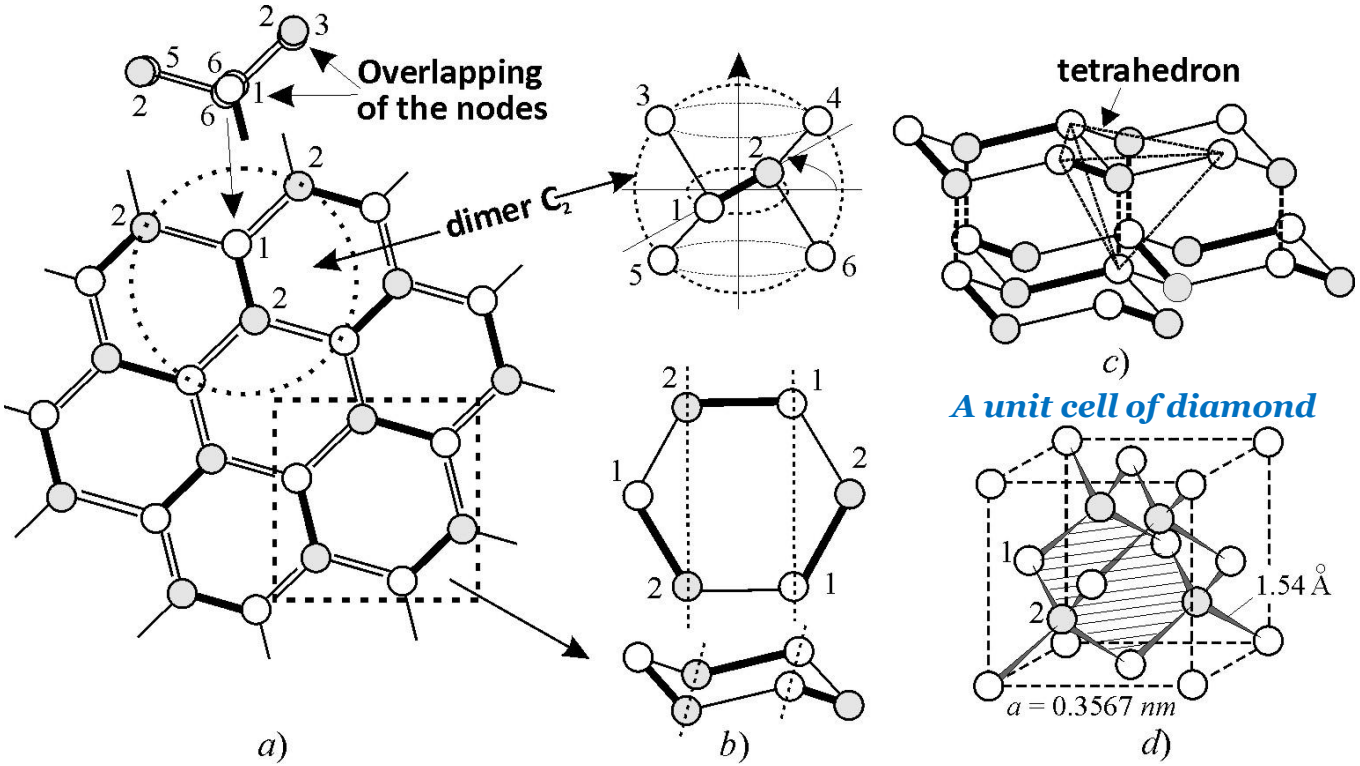
(*a*) The planar structure of carbon dimer C_2 (and carbon "atom" C).

(*b*) Azimuth state of nodes 1 and 2 of the inner shell, rotated by a phase angle $\alpha = \pi/4$, in relation to the outer shell, allowed by the solution for φ, since $\Phi_p(\varphi) = \Phi_m \cos(m\varphi + \alpha)$.

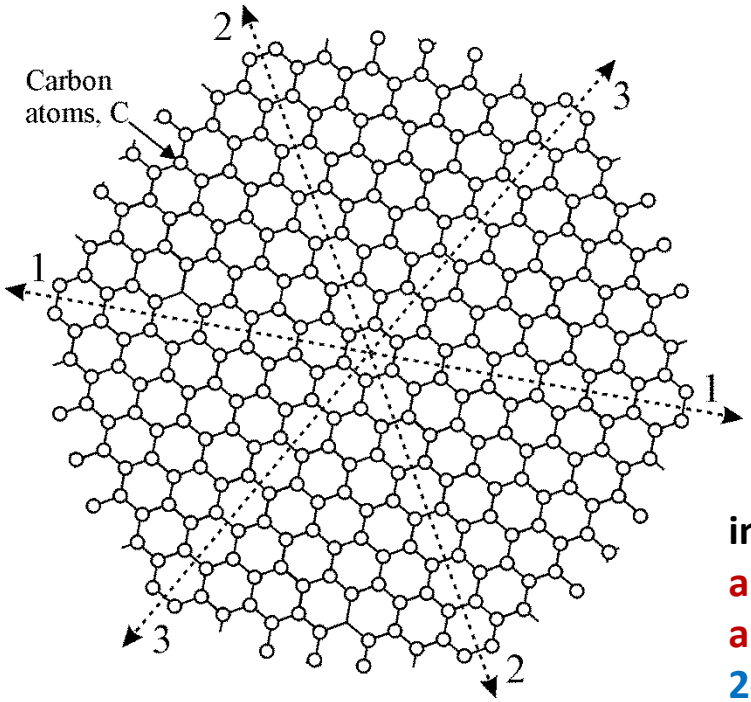
(*c*) Formation of bonds (dashed lines) between rotated internal nodes, 1 and 2, adjacent dimmers of carbon , which results in a face-centered cubic structure of diamond.

Face-centered cubic lattice of diamond

(consisting of C_2 carbon dimers)



Traditional view at the basic properties of graphene



* [Robert E. Newnham, Properties of Materials: Anisotropy, Symmetry, Structure; Oxford University Press, 2005]

Graphene is an allotrope of carbon in the form of a two-dimensional hexagonal lattice.

Point group D6h, space group P6/mmm

- One atom is in the lattice site.
- sp^2 hybridized orbitals are responsible for the bond of carbon atoms in the lattice.
- The lattice has 6-Fold rotational symmetry.

Hence,

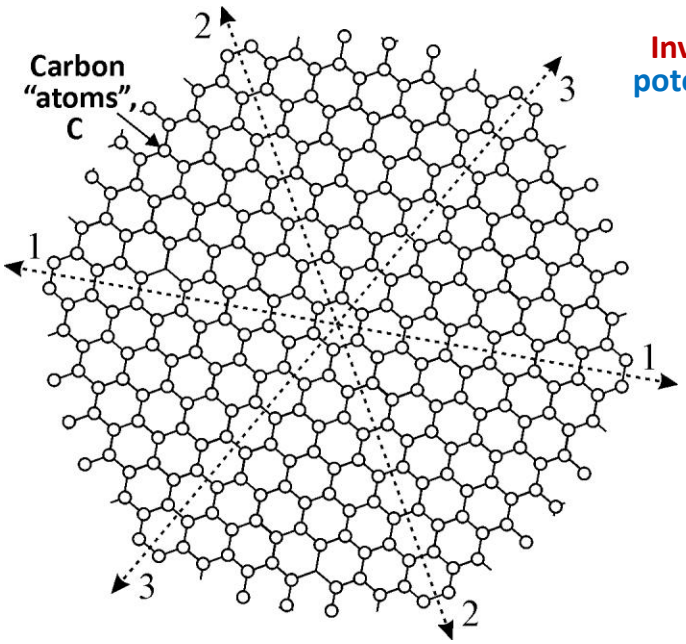
in full agreement with the basic symmetry theory*, all properties, including electronic conductivity, along the crystallographically identical directions 1-1, 2-2, 3-3 (indicated here) must be equal.

However, our studies show that it is not true

Two view on the structure of graphene

Conventional

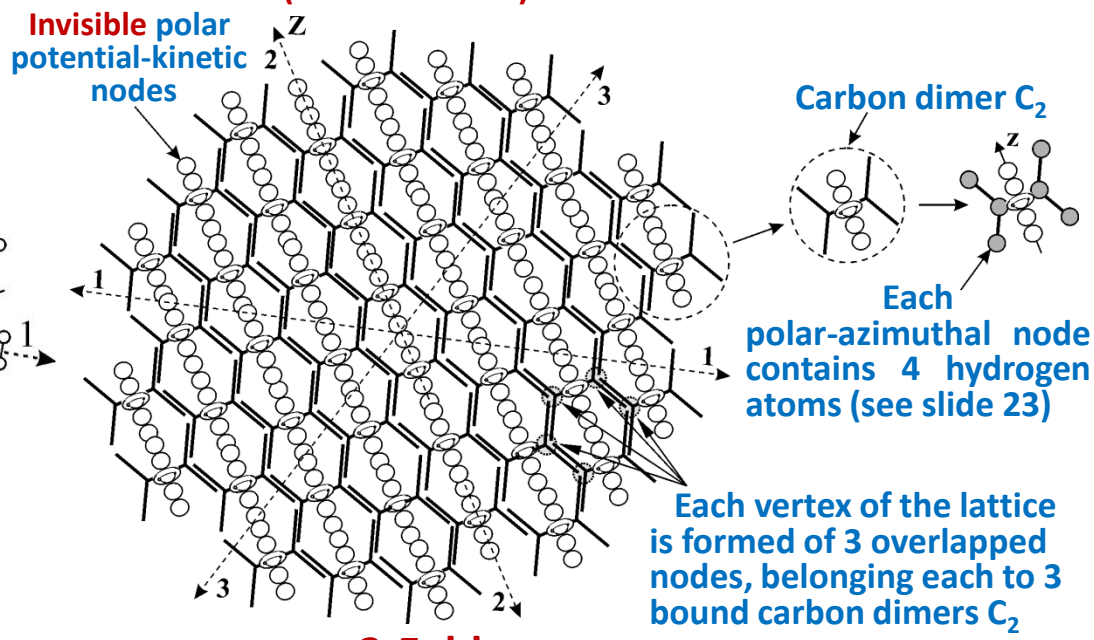
Hexagonal lattice is formed of ordinary carbon "atoms", C (with nucleus)



6-Fold
rotational symmetry

Predicted by solutions of the WM

Hexagonal lattice is formed of 6-node elementary molecules of H-atoms (without nuclei)



2-Fold
rotational symmetry

An ordered "covalent" bond of carbon dimers C_2 is realized in graphene along strong internodal bonds in such a way that a continuous chain of empty polar nodes forms hollow channels inside the crystal, through which charge carriers can move without obstacles not being scattered, like it occurs at the ballistic motion.

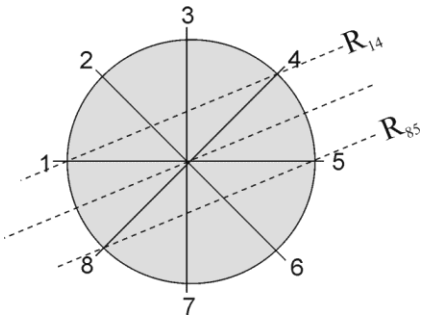
Which of the presented structures is closer to the real one?

The experiment showed that the structure of graphene formed from "atoms" with the shell-nodal structure is more adequate to reality.

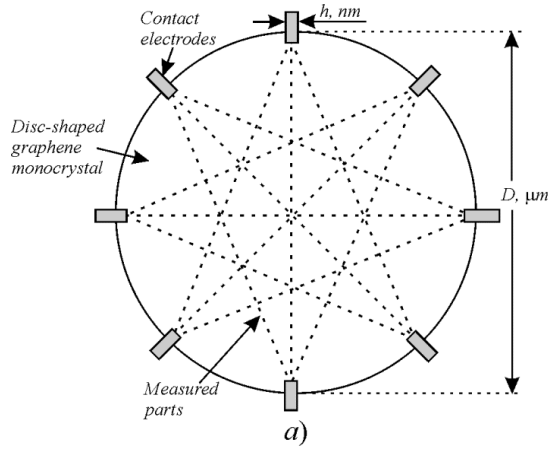
Experimental evidence

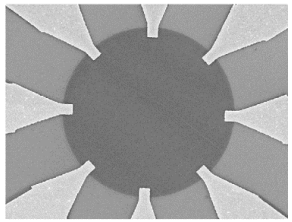
was obtained

by discovering the conduction anisotropy in graphene, which is naturally inherent in the structure shown above, following from the WM.



A pair of identical conductive sections





b)

Fig. Measurement scheme (*a*) and SEM image (*b*) of graphene round sheets during testing (conducted by us in 2010):

Both resistances between parallel pairs of contacts (for example, such as R_{14} and R_{85}), embracing identical conductive areas (i.e., having the same geometrical configuration), should be equal in magnitude (within the permissible error).

This allows to instantly monitor the perfection of electric contacts with disc-shaped graphene plates, directly in the process of measuring resistance.

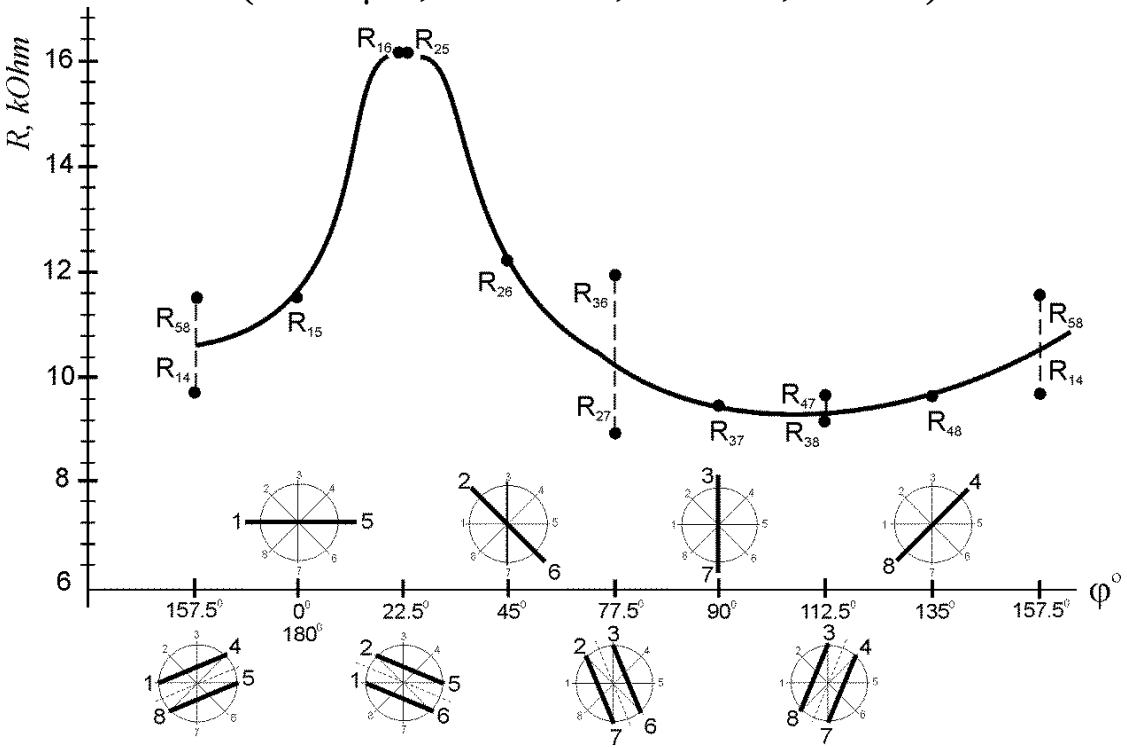
Therefore, the given scheme can be used to test the effectiveness of various technological methods developing for creating improved electric contacts with graphene.

Measurement results:

Angular dependence of electrical resistance $R = f(\varphi)$

in the plane of a graphene sheet of circular shape

(D=10 μm, h=580 nm, T=4.2 K, I=1 nA)



The obtained dependence is characteristic for anisotropic materials having two-fold symmetry

Anisotropic behavior of electrical conductivity in graphene

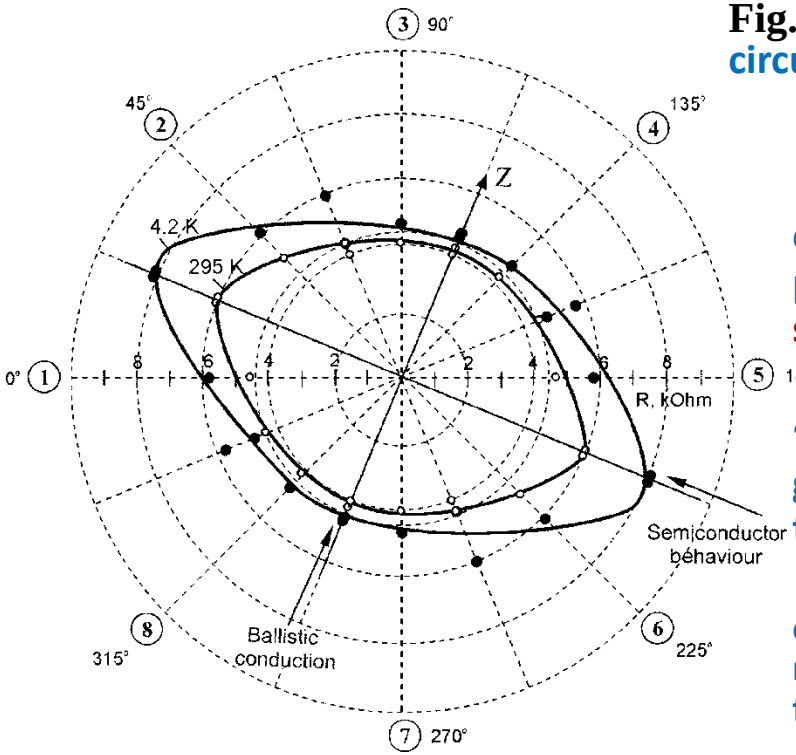


Fig. Polar diagram of resistance in the plane of a circular graphene sheet of monoatomic thickness:

(D=10 μm, h=580 nm, T=4.2 and 295 K)

Temperature dependence of graphene conductivity, along one of the directions in the plane, shows that graphene behaves like a semiconductor in this direction.

In the perpendicular direction, along the "ballistic channels" (z-axis), the resistance in graphene is practically independent of temperature; graphene behaves like metal.

(Comparison with metal refers only to the absence of a temperature dependence, because the conduction mechanisms in the metal and in graphene are fundamentally different)

Anisotropy of the hexagonal lattice of graphene, predicted by the author (2009), is the next in a series of the discoveries of the WM.

Together with other discoveries: the nature of all possible "atomic" isotopes and the primary cause of the Periodic Law, it is a direct experimental confirmation of the reality of the shell-nodal structure of "atoms".

Anisotropy

of

unstrained pristine graphene,

was subsequently

confirmed by optical methods:

1) Microscopic Reflection Difference Spectroscopy

in the visible-frequency range

(2014 - 2015, Beijing Key Laboratory).

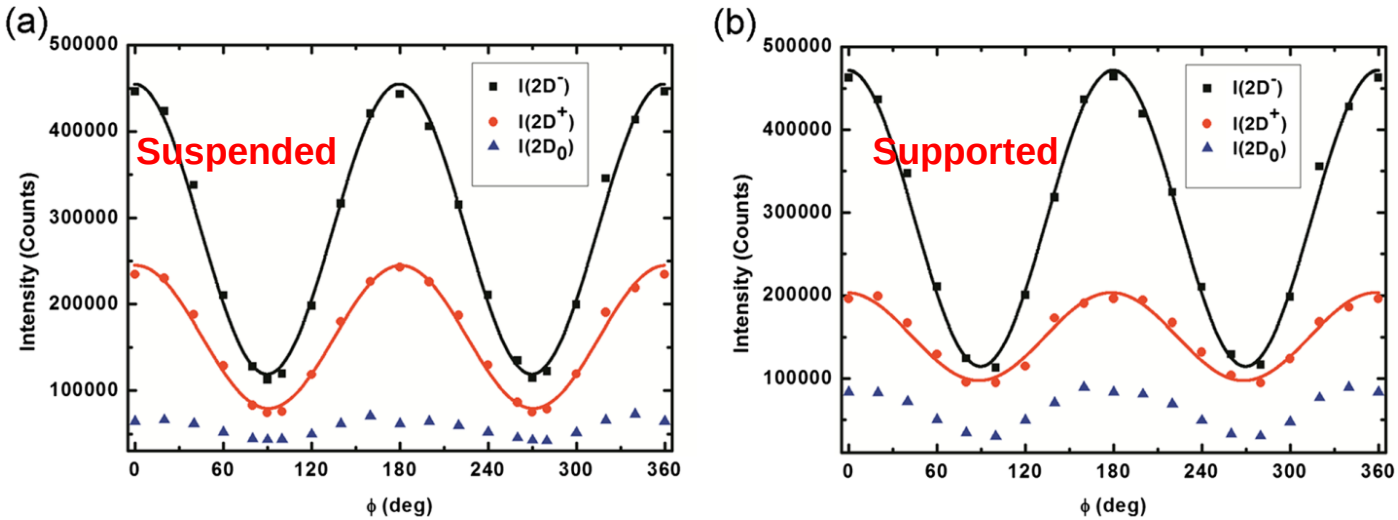
2) Polarized Raman spectroscopy

Also in absence of any external influences (unstrained from outside)

(2012, Taiwan), see below:

Strong sinusoidal intensity modulation*
with a period of 180°

The estimated peak positions: 2.647 and 2.660 (a), 2.647 and 2.660 cm^{-1} (b), respectively, for the $2D^+$ and $2D^-$ sub-bands.



"Figure 4. Analysis of intensity. (a) Suspended and (b) supported graphene. The plots of $I(2D^+)$, $I(2D^-)$, and $I(2D_0)$ as functions of Φ. The symbol 'I' denotes the intensity of the corresponding sub-peak obtained by fitting the related Raman spectrum with a triple-Lorentzian function. The obtained intensities are shown by the dots, which are fitted by the form of $A\cos^2(\Phi - \Phi_0)$ for $I(2D^+)$ and $I(2D^-)$, and of a constant of A, where A and Φ_0 are fitting parameters. The black and red lines display the fitting results"*.

*[Huang et al, "*Observation of strain effect on the suspended graphene by polarized Raman spectroscopy*", Nanoscale Research Letters 2012, 7:533]

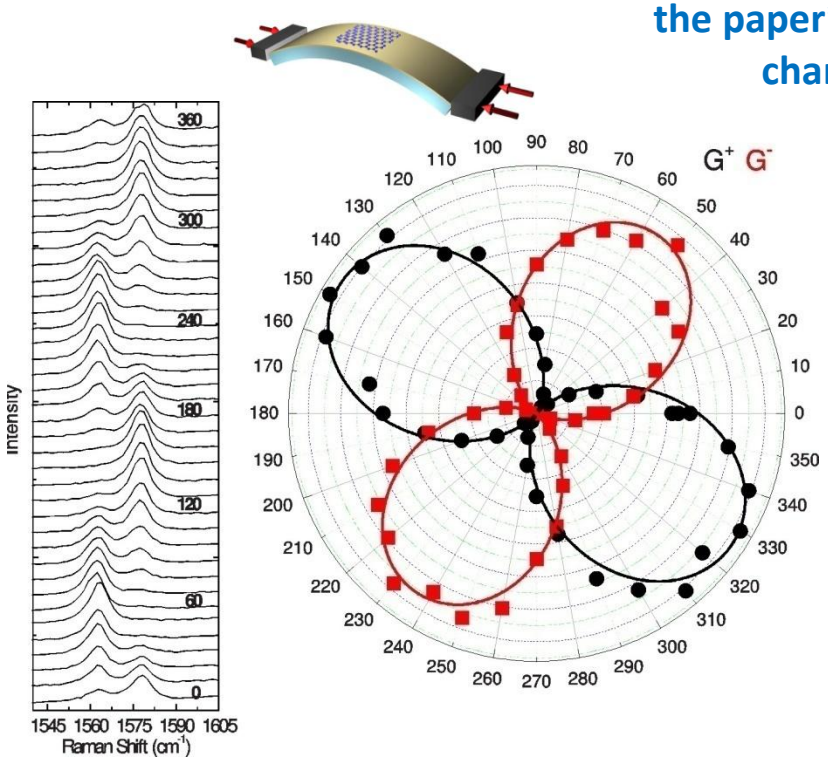
By now we have the grounds to argue that

Raman spectra

(presented in various papers on the influence of strains)

largely reflect the natural anisotropy inherent in unstrained graphene.

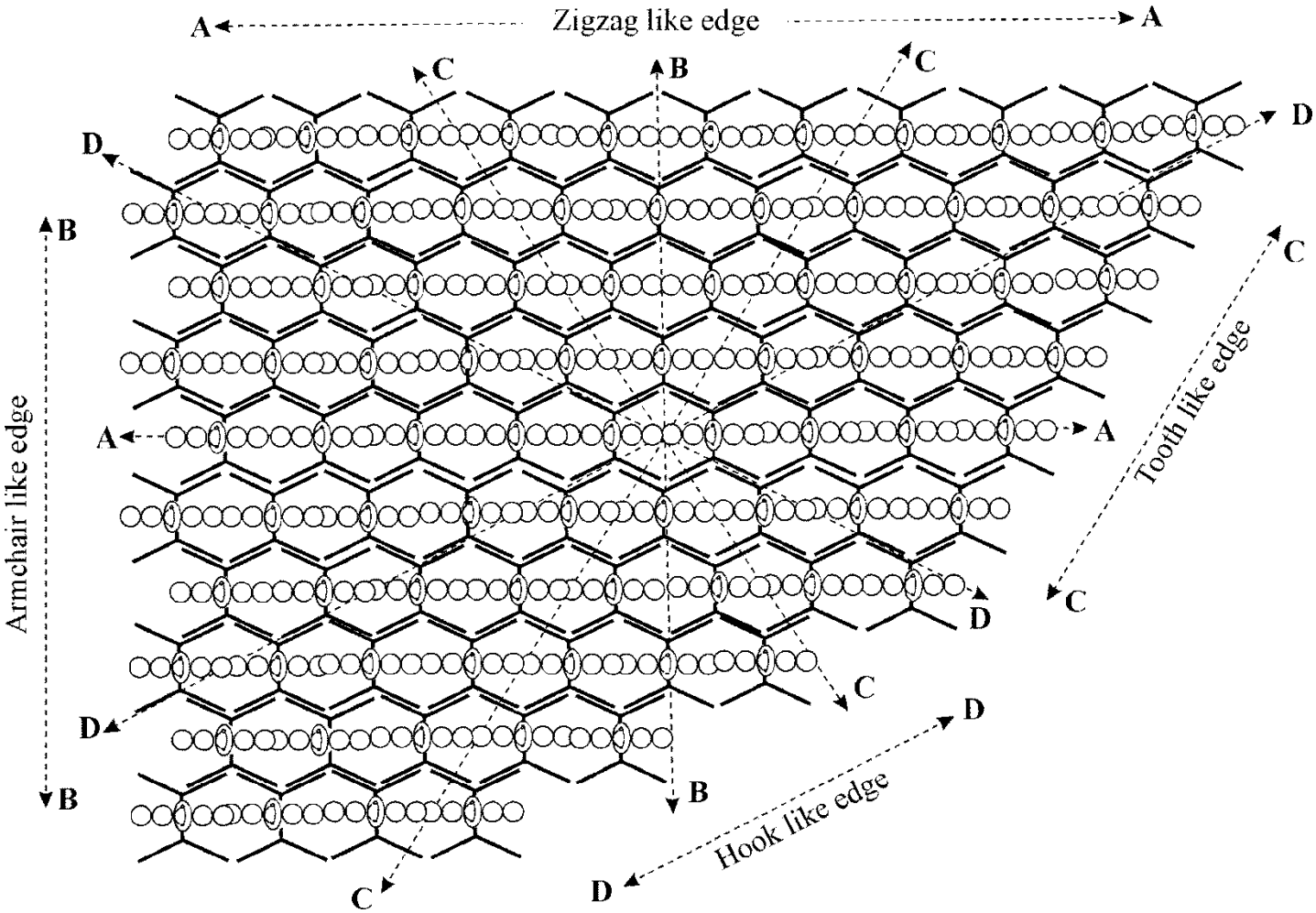
For example, strains, caused by bending of the hexagonal lattice of graphene (as in the paper* of 29.05.2009), only slightly distort the characteristic Raman spectra conditioned by anisotropic structure of graphene, not changing their main form (see Fig. 6).



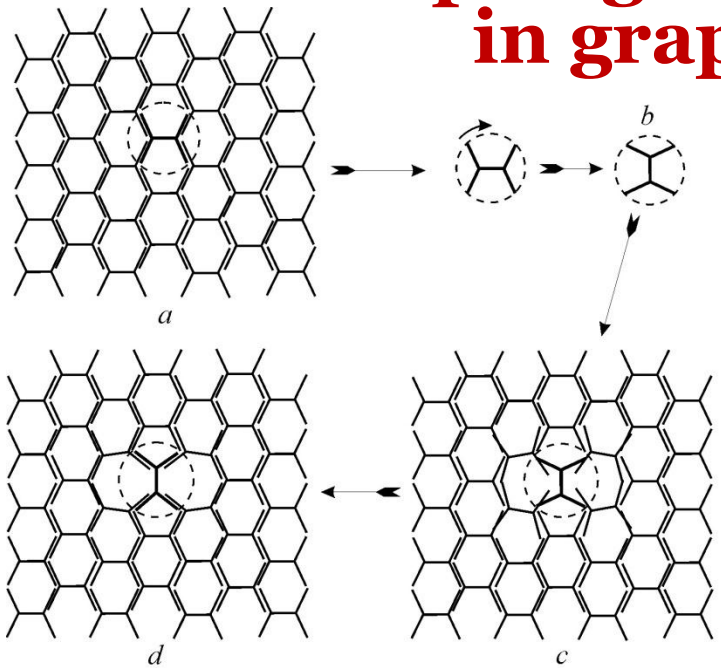
"Figure 6: (Color online) (Left) Raman spectra and (right) polar plot of the fitted G$^+$ and G$^-$ peaks as a function of the angle between the incident light polarization and the strain axis θ_{in} measured with an analyzer selecting scattered polarization along the strain axis, $\theta_{out} = 0$. The polar data are fitted to $I_{G^-} \propto \sin^2(\theta_{in} + 34^\circ)$ and $I_{G^+} \propto \cos^2(\theta_{in} + 34^\circ)$, see text".

*[T. M. G. Mohiuddin at al, "*Uniaxial strain in graphene by Raman spectroscopy: G peak splitting, Grüneisen parameters, and sample orientation*", Phys. Rev. B 79, 205433 (29.05.2009)]

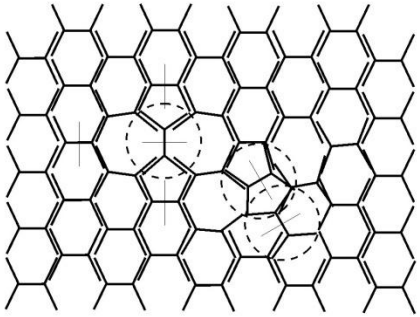
The shape of the edges in graphene



Topological defects in graphene



The formation of pentagon-heptagon defects (SW defects)

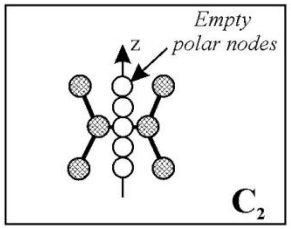


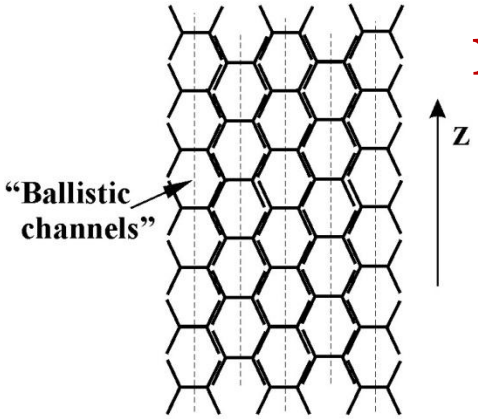
Scheme of binding in complex metastable defects

Topological defects, such as **pentagon-heptagon pairs,** appear due to arising, when heated, of the rotational modes (additionally to vibrational) that leads to a violation at some instant of equilibrium bonds and the formation of short-lived bonds with other nodes of the nearest carbon dimers, as indicated in this scheme.

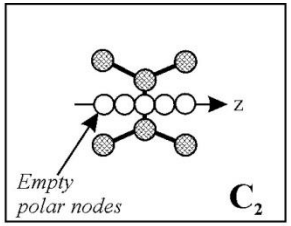
When cooling, **self-assembly** takes place: the excited region of the lattice returns to the initial state of equilibrium, the defect disappears.

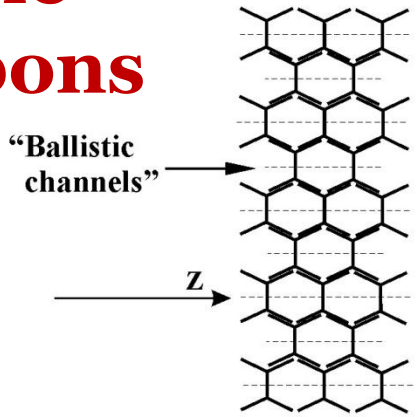
Graphene nanoribbons





Zigzag graphene nanoribbon
("ballistic" conductivity ↑)





Armchair graphene nanoribbon
(semicondicting behaviour ↑)

The shape of the edges of graphene nanoribbons, different widths and lengths, depends on the orientation of the crystallographic axis z in relation to the orientation of the edges.

Just a certain orientation of the z axis, but not the shape of the edges (as is commonly believed), affects the properties of nanoribbons, for example, electrical resistance, because the chains of empty polar nodes, which are responsible for the "ballistic" motion of charge carriers, are parallel to this axis (A-A, in the drawings).

We should not confuse cause and effect.

Now it becomes clear and understandable:

Why *"Graphene…is an interesting mix of a semiconductor…and a metal"*
Why *"Electrons in graphene … have very long mean free paths"*

[A.H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, *The electronic properties of graphene*, Rev. Mod. Phys. 81, 2009]

Why "*carbon nanotubes*, *rolled-up form of graphene, have either conductivity, metallic or semiconducting*".

[Tsuneya Ando, *The electronic properties of graphene and carbon nanotubes*, Review, NPG Asia Materials (2009) 1, 17–21; doi:10.1038/asiamat.2009.1]

In accordance with the WM, the rolling-up of graphene, resulting in the formation of nanotubes, is realized mainly along two crystallographic directions: along the major axis of anisotropy and in the direction perpendicular to it.

These two mutually perpendicular directions provide thermodynamically more stable states of a bonded system of carbon atoms in nanotubes, with minimum energy.

According to the Wave Model,

elementary characteristic directions of the probabilistic formation of material spaces are determined by the polar-azimuth functions

Therefore, it was natural to expect that the

characteristic angles of the functions

should be materialized in the

characteristic angles of the crystals.

Really,

as we have found, the numerical values of the angles of crystal faces directly follow from the solutions for the

polar component $\Theta_{l,m}(\theta)$

of the wave equation $\Delta\hat{\psi} + k^2\hat{\psi} = 0$, where

$$\hat{\psi} = f(\rho, \theta, \varphi) = A\hat{R}_l(\rho)\Theta_{l,m}(\theta)\hat{\Phi}_m(\varphi)$$

An equation for the polar constituent $\Theta_{l,m}(\theta)$

of the wave equation has the form,

$$\frac{d^2\Theta_{l,m}}{d\theta^2} + ctg\vartheta\frac{d\Theta_{l,m}}{d\theta} + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta_{l,m} = 0 \qquad (11)$$

Elementary solutions of the polar equation are as follows,

$$\Theta_{l,m}(\theta) = C_{l,m}\cdot P_{l,m}(\cos\theta) \qquad (12)$$

where $C_{l,m}$ is the coefficient depending on the normalization conditions, and $P_{l,m}(\cos\theta)$ are Legendre adjoined functions:

$$P_{l,m}(\cos\theta) = \frac{\sin^m\theta}{2^l\, l!}\frac{d^{l+m}}{d(\cos\theta)^{l+m}}\left(\cos^2\theta - 1\right)^l \qquad (13)$$

The normalization of the polar component is determined by the condition:

$$\int_0^\pi \left|\Theta(\theta)\right|^2 \sin\theta d\theta = 1 \qquad (14)$$

The normalized constant $C_{l,m}^0$ for the polar component (12) is

$$C_{l,m} = \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}} \qquad (15)$$

Table 1

Polar components $\Theta_{l,m}(\theta)$ (12) of the solution of the equation (11) **with the normalizing factor** $C_{l,m}$ (15) **are presented** in Table 1.

l	m	$\Theta_{l,m}(\theta)$
0	0	$\sqrt{2}/2$
1	0	$(\sqrt{6}/2)\cos\theta$
	1	$(\sqrt{3}/2)\sin\theta$
2	0	$(\sqrt{10}/4)(3\cos^2\theta - 1)$
	1±	$(\sqrt{15}/2)\sin\theta\cos\theta$
	2±	$(\sqrt{15}/4)\sin^2\theta$
3	0	$(\sqrt{14}/4)\cos\theta\,(5\cos^2\theta - 3)$
	1±	$(\sqrt{42}/8)\sin\theta\,(5\cos^2\theta - 1)$
	2±	$(\sqrt{105}/4)\sin^2\theta\cos\theta$
	3±	$(\sqrt{70}/8)\sin^3\theta$
4	0	$3\sqrt{2}/16 * (35\cos^4\theta - 30\cos^2\theta + 3)$
	1±	$3\sqrt{10}/8 * \sin\theta\cos\theta\,(7\cos^2\theta - 3)$
	2±	$3\sqrt{5}/8 * \sin^2\theta\,(7\cos^2\theta - 1)$
	3±	$3\sqrt{70}/8 * \sin^3\theta\cos\theta$
	4±	$3\sqrt{35}/16 * \sin^4\theta$
5	0	$\sqrt{22}/16 * \cos\theta\,(63\cos^4\theta - 70\cos^2\theta + 15)$
	1±	$\sqrt{165}/16 * \sin\theta\,(21\cos^4\theta - 14\cos^2\theta + 1)$
	2±	$\sqrt{1155}/8 * \sin^2\theta\cos\theta\,(3\cos^2\theta - 1)$
	3±	$\sqrt{770}/16 * \sin^3\theta\,(9\cos^2\theta - 1)$
	4±	$3\sqrt{385}/32 * \sin^4\theta\cos\theta$
	5±	$3\sqrt{154}/32 * \sin^5\theta$

Characteristic angles

of the polar functions $\Theta_{l,m}(\theta)$ are zeros and extremal values.

To calculate them, it is convenient to use the reduced form $\tilde{\Theta}_{l,m}(\theta)$ of these functions presented in Table 2.

l	m	$\tilde{\Theta}_{l,m}(\theta)$
0	0	1
1	0	$\cos\theta$
	1	$\sin\theta$
2	0	$\cos^2\theta - 1/3$
	±1	$\sin\theta\cos\theta$
	±2	$\sin^2\theta$
3	0	$\cos\theta(\cos^2\theta - 3/5)$
	±1	$\sin\theta(\cos^2\theta - 1/5)$
	±2	$\sin^2\theta\cos\theta$
	±3	$\sin^3\theta$
4	0	$\cos^4\theta - 6/7\cos^2\theta + 3/35$
	±1	$\sin\theta\cos\theta(\cos^2\theta - 3/7)$
	±2	$\sin^2\theta(\cos^2\theta - 1/7)$
	±3	$\sin^3\theta\cos\theta$
	±4	$\sin^4\theta$

Table 2

l	m	$\tilde{\Theta}_{l,m}(\theta)$
	±1	$\sin\theta(\cos^4\theta - 2/3\cos^2\theta + 1/21)$
	±2	$\sin^2\theta\cos\theta(\cos^2\theta - 1/3)$
	±3	$\sin^3\theta(\cos^2\theta - 1/9)$
	±4	$\sin^4\theta\cos\theta$
	±5	$\sin^5\theta$
6	0	$\cos^6\theta - 15/11\cos^4\theta + 5/11\cos^2\theta - 5/231$
	±1	$\sin\theta\cos\theta(\cos^4\theta - 10/11\cos^2\theta + 5/33)$
	±2	$\sin^2\theta(\cos^4\theta - 6/11\cos^2\theta + 1/33)$
	±3	$\sin^3\theta\cos\theta(\cos^2\theta - 3/11)$
	±4	$\sin^4\theta(\cos^2\theta - 1/11)$
	±5	$\sin^5\theta\cos\theta$
	±6	$\sin^6\theta$

Verification

of the correctness of our prediction about the wave nature of crystals was made [1, 7, 8] by comparing numerical values of zeros and extremal values of the angles following from solutions for the polar constituent $\tilde{\Theta}_{l,m}(\theta)$ of the wave equation (presented in Table 2), and their sums and differences, with experimental data for the angles of crystals known from the literature (compiled mainly by R. Haüy [9] and N. Kokscharov [10, 11]).

When comparing angles, we are not interested in the composition of minerals.

We denoted zeros of the polar functions $\tilde{\Theta}_{l,m}(\theta)$ by the symbol $O_s(l,m)$, where the subscript "s" indicates the number of the root.

Similarly, the angles of extreme values of $\tilde{\Theta}_{l,m}(\theta)$ are denoted as $\theta_s(l,m)$.

Obviously, every angle is characterized simultaneously by two measures: θ and $\pi - \theta$.

The comparison results

(taken from the Lectures of the author [12])

are as follows:

Characteristic angles of the solutions $\widetilde{\Theta}_{l,m}(\theta)$ and crystals of the natural minerals.

Characteristic angles of $\widetilde{\Theta}_{l,m}(\theta)$ (*theoretical* values first calculated and published by L. Kreidik and G. Shpenkov [2-4])	The angles of crystal minerals (*measured* by R. Haüy [5], N. Kokscharov [6, 7], and others [8-23])

$$\boxed{\widetilde{\Theta}_{2,0}(\theta) = \cos^2\theta - 1/3}$$

a) Zeros:

$O_1(2,0) = \arccos\sqrt{\frac{1}{3}} = 54^\circ\,44'\,8.''20$ — $54^\circ\,44'\,8.''20$

$O_2(2,0) = \arccos\left(-\sqrt{\frac{1}{3}}\right) = 125^\circ\,15'\,51.''80$ — $125^\circ 15' 52''$ [5: Part I, Vol. III, p.364, 1853]

b) Sectors:

$2O_1(2,0) = 109^\circ\,28'\,16.''40$ — Haüy: $109^\circ 28' 16''$ [5: p.29; 1*]

$O_2(2,0) - O_1(2,0) = 70^\circ\,31'\,43.''60$ — Haüy: $70^\circ 31' 44''$ [5: p.29; 1*]

$2(O_2(2,0) - O_1(2,0)) = 141^\circ\,03'\,27.''20$ — $141^\circ\,03'$ [6: Part. III, Vol. VII, p.26, 1844]

c) Extremes: $\theta_1(2,0) = 0^\circ$, $\theta_2(2,0) = 90^\circ$, and $\theta_3(2,0) = 180^\circ$ — characteristic angles of crystals

$$\boxed{\widetilde{\Theta}_{2,1} = \cos\theta\sin\theta}$$

Zeros and **extremes** of 45° and 90° — characteristic angles of crystals

$$\boxed{\widetilde{\Theta}_{3,0} = \cos\theta(\cos^2\theta - 3/5)}$$

a) Zeros:

$O_1(3,0) = \arccos\left(\sqrt{\frac{3}{5}}\right) = 39^\circ\,13'\,53.''47$ — Haüy: $39^\circ 13' 53''$ [5: p.85; 2*]

$O_2(3,0) = 90^\circ$,

$O_3(3,0) = \arccos\left(-\sqrt{\frac{3}{5}}\right) = 140^\circ\,46'\,06.''53$ — Haüy: $140^\circ 46' 7''$ [5: p.86; 2*]

b) Sectors:

$2O_1(3,0) = 78^\circ\,27'\,46.''94$ — Haüy: $78^\circ\,27'\,47''$ [5: p.85; 2*]

$O_2(3,0) - O_1(3,0) = 50^\circ\,46'06.''53$ — Haüy: $50^\circ 46' 06.''50$ [5: p.46; 3*]

$O_3(3,0) - O_1(3,0) = 101^\circ\,32'13.''06$ — Haüy: $101^\circ\,32'\,13''$ [5: p.46; 3*]

c) Extremes:

$\theta_1(3,0) = 0^\circ$

$\theta_2(3,0) = \arccos\left(\sqrt{\frac{1}{5}}\right) = 63^\circ\,26'\,05.''82$ — Haüy: $63^\circ\,26'\,06''$ [5: p.61; 4*]

$\theta_3(3,0) = \arccos\left(-\sqrt{\frac{1}{5}}\right) = 116^\circ\,33'\,54.''18$ — Haüy: $116^\circ\,33'\,54''$ [5: p.61; 4*]

$\theta_4(3,0) = 180^\circ$

d) Sectors:

$2\theta_2(3,0) = 126^\circ\,52'\,11.''64$ — $126^\circ\,52'12''$ [6: Part I, Vol. III, p.322, 1853]

$\theta_3(3,0) - \theta_1(3,0) = 58^\circ 54' 32.''65$ — $58^\circ\,57'22''$? [6: Part IV, Vol. XI, p.394, 1860]

$\theta_3(3,0) - \theta_2(3,0) = 53^\circ\,07'\,48.''36$ — Haüy: $53^\circ\,07'\,50''$ [5: p.58; 5*]

$2(\theta_3(3,0) - \theta_2(3,0)) = 106^\circ 15' 36.''72$ — $106^\circ\,17'26''$? [6: Part I, Vol. III, p.90, 1869]

$106^\circ\,12'$ [6: Part IV, Vol. XII, p.501,1866]

average value $106^\circ\,14'43''$

$360^\circ - 2(\theta_3(3,0) - \theta_2(3,0)) = 147^\circ\,28'\,46.''56$ — $147^\circ\,29'\,34''$ [6: ч. I, кн. III, 347, 1853]

$$\boxed{\widetilde{\Theta}_{3,1}(\theta) = \sin\theta(\cos^2\theta - 1/5)}$$

a) Zeros repeat extremes $\widetilde{\Theta}_{3,0}(\theta)$:

$O_1(3,1) = 0^\circ$, $O_2(3,1) = \arccos\left(\sqrt{\frac{1}{5}}\right)$, $O_3(3,1) = \arccos\left(-\sqrt{\frac{1}{5}}\right)$, $O_4(3,2) = 180^\circ$

b) Extremes:

$\theta_1(3,1) = \arccos\left(\sqrt{\frac{11}{15}}\right) = 31^\circ\,05'\,27.''35$ — $31^\circ 05' 06''$ [6: Part III, Vol. VII, p.99, 1869]

$31^\circ 06'$ [6: Part. III, Vol. VII, p.108, 1869]

average value $31^\circ 05' 33''$

$\theta_2(3,1) = 90^\circ$,

$\theta_3(3,1) = \arccos\left(-\sqrt{\frac{11}{15}}\right) = 148^\circ\,54'\,32.''65$ — $149^\circ\,02'11''$? [6: Part IV, Vol. XI, p.283, 1866]

c) Sectors:

$2\theta_1(3,1) = 62^\circ\,10'\,54.''70$ — $62^\circ\,12'54''$? [6: Part IV, Vol. XI, p.390, 1860]

$180^\circ - 2\theta_1(3,1) = 117^\circ\,49'\,5.''30$ — $117^\circ\,48'43''$ [6: Vol. IV, p.110, 1870]

$\theta_2(3,1) - \theta_1(3,1) = 58^\circ\,54'\,32.''65$ — $58^\circ\,57'22$? [6: Part IV, Vol. XI, p.394, 1860]

$180^\circ - (\theta_2(3,1) - \theta_1(3,1)) = 121^\circ\,05'\,27.''35$ — $121^\circ 05'$ [6: Part I, Vol. II, p.169, 1853]

$2(\theta_3(3,0) - \theta_2(3,0)) = 106^\circ 15' 36.''72$ — $106^\circ 17' 26''$? [6: Part I, Vol. III, p.90, 1869]

$$\boxed{\widetilde{\Theta}_{3,2}(\theta) = \sin^2\theta\cos\theta}$$

a) Zeros:

$O_1(3,2) = 0^\circ$, $O_2(3,2) = 90^\circ$ $O_3(3,2) = 180^\circ$

c) Extremes:

$\theta_1(3,2) = \arccos\left(\frac{1}{\sqrt{3}}\right) = 54^\circ\,44'8.''20$, $\theta_2(3,2) = \arccos\left(-\frac{1}{\sqrt{3}}\right) = 125^\circ 15'51.''80$

repeat zero $\Theta_{2,0}(\theta)$.

$$\boxed{\widetilde{\Theta}_{4,0}(\theta) = \cos^4\theta - \tfrac{6}{7}\cos^2\theta + \tfrac{3}{35}}$$

a) Zeros

$$O_1(4,0) = \arccos\left(\sqrt{\frac{3}{7} + \sqrt{\left(\frac{3}{7}\right)^2 - \frac{3}{35}}}\right) = 30^\circ\,33'20.''13$$

$30^\circ 32' 48''$ [6: Part IV, Vol. XI, p.392, 1860]

$$O_2(4,0) = \arccos\left(\sqrt{\frac{3}{7} - \sqrt{\left(\frac{3}{7}\right)^2 - \frac{3}{35}}}\right) = 70^\circ\,07'27.''41$$

$70^\circ 04' 33''$ [6: Vol. VII, p.117, 1870]

$70^\circ 10' 22''$ [6: Vol. V, p.303, 1870]

average value $70^\circ 7' 28$

$$O_3(4,0) = \arccos\left(-\sqrt{\frac{3}{7} - \sqrt{\left(\frac{3}{7}\right)^2 - \frac{3}{35}}}\right) = 109^\circ\,52'32.''.59$$

$109^\circ\,52' 00''$ [6: Part IV, Vol. XI, p.404, 1860]

$109^\circ\,53' 07''$ [6: Vol. IV, p.101, 1870]

average value $109^\circ\,52' 33''$

$$O_4(4,0) = \arccos\left(-\sqrt{\frac{3}{7} + \sqrt{\left(\frac{3}{7}\right)^2 - \frac{3}{35}}}\right) = 149^\circ\,26'39.''87$$

$149^\circ\,26' 22''$ [6: p.20]

$149^\circ\,30' 56''$? [6: Part IV, Vol. XI, p.262, 1866]

$149^\circ\,22' 50''$? [6: Part IV, Vol. XI, p.262, 1866]

Average value $149^\circ\,26' 43''$

b) Sectors

$2O_1(4,0) = 61^\circ\,06'40.''26$ $61^\circ\,06'\,55''$ [6: Vol. IX, p.495, 1870]

$2O_2(4,0) = 140^\circ\,14'54.''82$ Erofeev: $140^\circ\,17' 52''$? [8: p.296]

$140^\circ\,03' 35''$? [6: Vol. IV, p.110, 1870]

$140^\circ\,25' 20''$? [6: Vol. IV, p.102, 1870]

The average of the last two values $140^\circ\,14' 28''$

$O_2(4,0) - O_1(4,0) = 39^\circ 34'07.''28$ $39^\circ\,32' 33''$ [6: Part IV, Vol. XI, p.390, 1860]

$O_3(4,0) - O_2(4,0) = 39^\circ 45'05.''18$ $39^\circ\,45' 22''$ [6: Part IV, Vol. XI, p.392, 1860]

$2(O_4(4,0) - O_1(4,0)) - 180^\circ = 57^\circ 46'34.''56$ $57^\circ\,44' 25''$ [6: Part IV, Vol. XI, p.289, 1866]

$O_3(4,0) - O_1(4,0) = O_4(4,0) - O_2(4,0) = 79^\circ 19'12.''46$ $79^\circ\,18' 10''$ [6: Vol. IV, p.99, 1870]

$180^\circ - (O_3(4,0) - O_1(4,0)) = 100^\circ 40'47.''54$ $100^\circ\,41' 50''$ [6: Vol. IV, p. 111, 1870,]

$O_4(4,0) - O_1(4,0) = 118^\circ 53'19.''74$ $118^\circ\,53' 00''$ [6: Part III, Vol. VII, 46, 1853]

$118^\circ\,53' 50''$ [6: Vol. XI, p.479, 1870]

Average value $118^\circ\,53' 25''$

$360^\circ - 2(O_4(4,0) - O_1(4,0)) = 122^\circ 13'20.''52$ Glinka: $122^\circ 15'$ [9: 73]

$122^\circ 08' 29''$ [6: Part I, Vol. III, p.346, 1853]

$122^\circ 19$ [6: Part I, Vol. II, p.169, 1853]

The average of the last two values $122^\circ 13' 44''$

c) Extremes:

$\theta_1(4,0) = 0^\circ$,

$\theta_2(4,0) = \arccos\left(\sqrt{\frac{3}{7}}\right) = 49^\circ\,06'23.''78$ $49^\circ 12' 05''$? [6: Vol. IV, p.99, 1870]

$\theta_3(4,0) = 90^\circ$

$\theta_4(4,0) = \arccos\left(-\sqrt{\frac{3}{7}}\right) = 130^\circ\,53'36.''22$ $130^\circ 50'$ [6: Part III, Vol. VIII, p.306, 1855]

$\theta_5(4,0) = 180^\circ$

d) Sectors

$2\theta_2(4,0) = 360^\circ - 2\theta_4(4,0) = 98^\circ 12'47.''56$ $98^\circ 13' 48''$ [6: Part IV, Vol. XI, p.388, 1860]

$98^\circ 10' 55''$? [6: Vol. IV, p.110, 1870]

average value $98^\circ 12' 21''$

$\theta_3(4,0) - \theta_2(4,0) = 40^\circ 53'36.''22$ $40^\circ 49' 18''$? [6: Vol. II, N. 6, p.308, 1878]

$\theta_4(4,0) - \theta_2(4,0) = 81^\circ 47'12.''44$ $81^\circ 47' 00''$ [6: Part IV, Vol. XI, p.415, 1860]

$180^\circ - (\theta_3(4,0) - \theta_2(4,0)) = 139^\circ 06'23.''78$ $138^\circ 59' 41''$? [6, Vol. IV, p.102, 1870]

$$\boxed{\widetilde{\Theta}_{4,1}(\theta) = \sin\theta\cos\theta(\cos^2\theta - \tfrac{3}{7})}$$

a) Zeros

$O_1(4,1) = 0^\circ$,

$O_2(4,1) = \arccos\sqrt{\frac{3}{7}} = 49^\circ\, 6'\, 23.''20$ — $49^\circ 10' 42''$? [6: Vol. II, N.6, p.309, 1878]

$O_3(4,1) = 90^\circ$,

$O_4(4,1) = \arccos\left(-\sqrt{\frac{3}{7}}\right) = 130^\circ\, 53'\, 36.''22$ — $130^\circ 50'$? [6: Part III, Vol. VIII, p.306, 1855]

$O_5(4,1) = 180^\circ$

b) Sectors

$2O_2(4,1) = 360^\circ - 2O_4(4,1) = 98^\circ 12' 47.''56$ — $98^\circ 13' 48''$ [6: Part IV, Vol. XI, p.388, 1860]

$98^\circ 10' 55''$? [6: Vol. IV, p.110, 1870]

Average value $98^\circ 12' 21''$

$O_3(4,1) - O_2(4,1) = 40^\circ 53' 36.''22$ — $40^\circ 49' 18''$? [6: Vol. II, N.6, p.308, 1878]

$O_4(4,1) - O_2(4,1) = 81^\circ 47' 12.''44$ — $81^\circ 47' 0''$ [6: Part IV, Vol. XI, p.415, 1860]

$180^\circ - (O_3(4,1) - O_2(4,1)) = 139^\circ 06' 23.''78$ — $138^\circ 59' 41''$? [6: Vol. IV, p.102, 1870]

c) Extremes:

$$\theta_1(4,1) = \arccos\left(\sqrt{\frac{27}{56} + \sqrt{\left(\frac{27}{56}\right)^2 - \frac{3}{28}}}\right) = 23^\circ\, 52'\, 40.''17$$

$23^\circ 59' 46''$? [6: Part III, Vol. VII, p.71, 1869]

$$\theta_2(4,1) = \arccos\left(\sqrt{\frac{27}{56} - \sqrt{\left(\frac{27}{56}\right)^2 - \frac{3}{28}}}\right) = 69^\circ\, 01'\, 29.''07$$

$69^\circ 01' 59''$ [6: Part IV, Vol. XI, p.392, 1860]

$69^\circ 01' 00''$ [6: Vol. IV, p.264, 1870]

Average value $69^\circ 01' 29''$

$$\theta_3(4,1) = \arccos\left(-\sqrt{\frac{27}{56} - \sqrt{\left(\frac{27}{56}\right)^2 - \frac{3}{28}}}\right) = 110^\circ\, 58'\, 30.''93$$

Glinka: $110^\circ 59' 45''$ [10: p.91]

$$\theta_4(4,1) = \arccos\left(-\sqrt{\frac{27}{56} + \sqrt{\left(\frac{27}{56}\right)^2 - \frac{3}{28}}}\right) = 156^\circ\, 07'\, 19.''83$$

$156^\circ 06' 00''$ [6, Part IV, Vol. XI, p.392, 1860]

$155^\circ 58' 21''$ [6, Vol. II, N.6, p.326, 1878]

Lebedev: $156^\circ 17' 11''$? [11: p.278]

Average $156^\circ 06' 53''$

d) Sectors

$2\theta_1(4,1) = 47^\circ 45' 20.''34$ — $47^\circ 45' 15''$ [6: Part IV, Vol. XI, p.287, 1866]

$180^\circ - 2\theta_1(4,1) = \theta_4(4,1) - \theta_1(4,1) = 132^\circ 14' 39.''66$ — $132^\circ 14' 45''$ [6: Part IV, Vol. XI, p.279, 1866]

$2\theta_2(4,0) = 2(180^\circ - \theta_3(4,0)) = 138^\circ 02' 58.''14$ — $138^\circ 06' 15''$? [6: Vol. II, N.6, p.326,1878]

$137^\circ 56' 00''$? [6: Part IV, Vol. XI, p.424, 1860]

Average value $138^\circ 01' 08''$

$180^\circ - 2\theta_2(4,1) = \theta_3(4,1) - \theta_2(4,1) = 41^\circ 57' 01.''86$ — $42^\circ 01' 13''$? [6: Part IV, Vol. XI, p.285, 1866]

$2(\theta_3(4,1) - \theta_2(4,1)) = 83^\circ 54' 3.''72$ — $83^\circ 54' 40''$ [6: Part III, Vol. VII, p.86, 1869]

$180^\circ - 2(\theta_3(4,1) - \theta_2(4,1)) = 96^\circ 05' 56.''28$ — $96^\circ 04' 11''$ [6: Vol. IV, p.111, 1870]

$$\boxed{\widetilde{\Theta}_{4,2}(\theta) = \sin^2\theta(\cos^2\theta - 1/7)}$$

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a) Zeros:

$O_1(4,2) = 0^\circ$,

$O_2(4,2) = \arccos\left(\frac{1}{\sqrt{7}}\right) = 67^\circ 47' 32.''44$ — $67^\circ 47' 30''$ [6: Vol. VII, p.261, 1870]

$O_3(4,2) = \arccos\left(-\frac{1}{\sqrt{7}}\right) = 112^\circ 12' 27.''56$ — $112^\circ 12' 40''$ [6: Vol. VII, p.267, 1870]

$112^\circ 12' 00''$ [6: Vol. VIII, p.261, 1870]

Average value $112^\circ 12' 20''$

$O_4(4,2) = 180^\circ$.

b) Sectors

$2O_2(4,2) = 135^\circ 35' 04.''88$ — $135^\circ 35' 30''$ [6: Part IV, Vol. XI, p.382, 1860]

$O_3(4,2) - O_2(4,2) = 44^\circ 24' 55.''11$ — Goldschmidt: $44^\circ 30' 30''$? [12: **9**, p.135, 1923]

$44^\circ 11' 19''$? [6: Part IV, Vol. XI, p.286, 1866]

$44^\circ 40' 04''$? [6: Part IV, Vol. XI, p.86, 1866]

Average value $44^\circ 25' 42''$

c) Extremes:

$\theta_1(4,2) - \arccos\left(\frac{2}{\sqrt{7}}\right) = 40^\circ 53' 36.''22$ — $40^\circ 49' 18''$? [6: Vol. II, N.6, p.368, 1878]

$\theta_2(4,2) = 90^\circ$

$\theta_3(4,2) - \arccos\left(-\frac{2}{\sqrt{7}}\right) = 139^\circ 6' 23.''78$ — $138^\circ 59' 41''$? [6: Vol. IV, p.102, 1870]

d) Sectors:

$2\theta_1(4,2) = 81^\circ 47' 12.''44$ — $81^\circ 47' 13''$ [6: Part. IV, Vol. XI, p.279, 1866]

$\theta_3(4,2) - \theta_1(4,2) = 98^\circ 12' 47.''56$ — $98^\circ 13' 48''$ [6: Part. IV, Vol. XI, p.388, 1860]

$98^\circ 10' 55''$ [6: Vol. IV, p.110, 1870], Average value $98^\circ 12' 21''$

$$\boxed{\widetilde{\Theta}_{4,3}(\theta) = \sin^3\theta\cos\theta}$$

Zeros: $O_1(4,3) = 0^\circ$, $O_2(4,3) = 90^\circ$, $O_3(4,3) = 180^\circ$, and
extremes: $\theta_1(4,3) = 60^\circ$, $\theta_2(4,3) = 120^\circ$ typical angles of crystals.

$$\boxed{\widetilde{\Theta}_{5,0}(\theta) = \cos\theta(\cos^4\theta - \tfrac{10}{9}\cos^2\theta + \tfrac{5}{21})}$$

a) Zeros:

$O_1(5,0) = \arccos\left(\sqrt{\frac{5}{9} + \sqrt{\left(\frac{5}{9}\right)^2 - \frac{5}{21}}}\right) = 25^\circ 01' 02.''42$ — $25^\circ 00' 15''$ [6: Part. IV, Vol. XI, p.435, 1860]

$O_2(5,0) = \arccos\left(\sqrt{\frac{5}{9} - \sqrt{\left(\frac{5}{9}\right)^2 - \frac{5}{21}}}\right) = 57^\circ 25' 13.''80$ — $57^\circ 30' 37''$? [6: Part. I, Vol. II, p.190, 1853]

$O_3(5,0) = 90^\circ$

$O_5(5,0) = \arccos\left(-\sqrt{\frac{5}{9} + \sqrt{\left(\frac{5}{9}\right)^2 - \frac{5}{21}}}\right) = 154^\circ 58' 57.''58$ — $154^\circ 49' 39''$? [5: p.90]

b) Sectors:

$O_1(5,0) = 50^\circ 02' 0\ \ 4.''84$ — $50^\circ 03' 35''$ [6: Vol. III, p.99, 1870]

$2O_2(5,0) = 114^\circ 50' 27.''60$ — $114^\circ 50'$ [6: Part. III, Vol. VIII, p.305, 1855]

$180^\circ - 2O_2(5,0) = O_5(5,0) - O_1(5,0) = 129^\circ 57'\ \ 55.''16$ — $129^\circ 56'$ [6: Vol. III, p.438, 1870]

$129^\circ 59' 58''$ [6: Vol. III, p.492, 1870]

$129^\circ 58' 15''$ [6: Part. II, Vol. IV, p.47, 1857]

Average value $129^\circ 58' 4''$

$O_5(5,0) - O_3(5,0) = 64^\circ 58' 57.''58$ — $64^\circ 58' 46''$ [6: Part. I, Vol. II, p.190, 1853]

$180^\circ - (O_5(5,0) - O_2(5,0)) = 82^\circ 26' 16.''22$ — $82^\circ 28' 56''$? [6: Vol. IV, p.431, 1870]

$O_5(5,0) - O_2(5,0) = 97^\circ 33' 43.''78$ — Kokscharov-son: $97^\circ 38' 24''$ [6: Vol. IV, N.11, p.223, 1879]

$97^\circ 27' 49''$ [6: Vol. IV, N.11, p.256, 1879]

Average value $97^\circ 33' 6''$

c) Extremes:

$\theta_1(5,0) = 0^\circ$

$\theta_2(5,0) = \arccos\left(\sqrt{\frac{1}{3} + \sqrt{\left(\frac{1}{3}\right)^2 - \frac{1}{21}}}\right) = 40^\circ 05' 17.''11$ Average value: $40^\circ 05' 0''$ [6: Vol. IV, N.11, p.349, 1879]

$\theta_3(5,0) = \arccos\left(\sqrt{\frac{1}{3} - \sqrt{\left(\frac{1}{3}\right)^2 - \frac{1}{21}}}\right) = 73^\circ 25' 38.''32$ — $73^\circ 30' 56''$? [6: Part. IV, Vol. XI, p.286, 1866]

$73^\circ 12' 14''$? [6: Vol. IV, p.113, 1870]

Average value $73^\circ 21' 35''$?

$\theta_4(5,0) = \arccos\left(-\sqrt{\frac{1}{3} - \sqrt{\left(\frac{1}{3}\right)^2 - \frac{1}{21}}}\right) = 106^\circ 34' 21.''68$ — $106^\circ 34'$ [10: p.90]

$106^\circ 27' 30''$? [6: Vol. II, N.6, p.318, 1878]

$106^\circ 38' 0''$? [6: Vol. II, N.6, p.318, 1878]

The average value of the two last angles: $106^\circ 32' 45'$?

$\theta_5(5,0) = \arccos\left(-\sqrt{\frac{1}{3} + \sqrt{\left(\frac{1}{3}\right)^2 - \frac{1}{21}}}\right) = 139^\circ 54' 42.''89$

Kokscharov-son: $139^\circ 55' 0''$ [6: Vol. IV, N.11, p.348, 1879]

$\theta_6(5,0) = 180^\circ$

d) Sectors:

$2\theta_2(5,0) = 80^\circ 10' 34.''22$ — $80^\circ 10' 15''$ [6: Part. I, Vol. III, p.343, 1853]

$2\theta_3(5,0) = 360^\circ - 2\theta_4(5,0) = 146^\circ 51' 16.''64$ — $146^\circ 48' 20''$? [6: Part. IV, Vol. XI, p.406, 1860]

Lebedev: $146^\circ 57' 00''$? [11: p.271];

Average value $146^\circ 52' 40''$

$180^\circ - 2\theta_2(5,0) = \theta_5(5,0) - \theta_2(5,0) = 99^\circ 49' 25.''78$ — $99^\circ 50' 00''$ [6: Part. I, Vol. III, p.343, 1853]

$\theta_3(5,0) - \theta_2(5,0) = 33^\circ 20' 21.''21$ — $33^\circ 17' 18''$? [6: Part. IV, Vol. XII, p.641, 1860]

$\theta_4(5,0) - \theta_2(5,0) = \theta_5(5,0) - \theta_3(5,0) = 66^\circ 29' 04.''57$ — $66^\circ 30' 08''$ [6: Part. IV, Vol. XII, p.641, 1860]

$\theta_4(5,0) - \theta_3(5,0) = 33^\circ 08' 43.''36$ — $33^\circ 04' 16''$? [6: Vol. IV, p.100, 1870]

$33^\circ 13' 21''$? [6: Vol. III, p.436, 1870]

Average value $33^\circ 08' 48''$

$$\boxed{\widetilde{\Theta}_{5,1}(\theta) = \sin\theta(\cos^4\theta - \tfrac{2}{3}\cos^2\theta + \tfrac{1}{21})}$$

a) Zeros repeat extremes $\widetilde{\Theta}_{5,0}(\theta)$.

b) Extremes:

$\theta_1(5,1) = \arccos\left(\sqrt{\frac{3}{5} + \sqrt{\left(\frac{3}{5}\right)^2 - \frac{29}{105}}}\right) = 19^\circ 24' 56.''02$ — $19^\circ 23' 03''$ [6: Part. I, Vol. VII, p.68, 1869]

$\theta_2(5,1) = \arccos\left(\sqrt{\frac{3}{5} - \sqrt{\left(\frac{3}{5}\right)^2 - \frac{29}{105}}}\right) = 56^\circ 08' 08.''88$ — $56^\circ 04' 57''$? [6: Vol. I, N.1, p.104, 1877]

$\theta_3(5,1) = 90^\circ$

$\theta_4(5,1) = \arccos\left(-\sqrt{\frac{3}{5} - \sqrt{\left(\frac{3}{5}\right)^2 - \frac{29}{105}}}\right) = 123^\circ 51' 51.''12$ — $123^\circ 53' 13''$? [6: Vol. IV, p.107, 1870]

$\theta_5(5,1) = \arccos\left(-\sqrt{\frac{3}{5} + \sqrt{\left(\frac{3}{5}\right)^2 - \frac{29}{105}}}\right) = 160^\circ 35' 03.''98$ — $160^\circ 34' 29''$ [6: Vol. IV, p.107, 1870]

c) Sectors:

$2\theta_1(5,1) = 38^\circ 49' 52.''04$ — $38^\circ 43' 53''$? [6: Part. IV, Vol. XI, p.286, 1866]

$38^\circ 51' 32''$? [6: Part. IV, Vol. XII, p.630, 1860]

Average value $38^\circ 47' 42''$

$2\theta_2(5,1) = 112^\circ 16' 17.''76$ — $112^\circ 16' 40''$ [6: Vol. VIII, p.255, 1870]

$\theta_4(5,1) - \theta_2(5,1) = 180^\circ - 2\theta_2(5,1) = 67^\circ 43' 42.''24$ — $67^\circ 43' 40''$ [6: Vol. VIII, p.261, 1870]

$180^\circ - 2(\theta_4(5,1) - \theta_2(5,1)) = 44^\circ 32' 35.''52$ — $44^\circ 40' 04''$? [6: Part. III, Vol. VII, p.86, 1869]

$\Delta\theta_5 = \theta_3(5,1) - \theta_1(5,1) = 70^\circ 35' 3.''98$ — $70^\circ 35' 00''$ [6: Part. IV, Vol. XI, p.269, 1866]

$180^\circ - (\theta_3(5,1) - \theta_1(5,1)) = 109^\circ 24' 56.''02$ — $109^\circ 24' 52''$ [6: Part. IV, Vol. XI, p.386, 1860]

$360^\circ - 2(\theta_5(5,1) - \theta_1(5,1)) = 77^\circ 39' 44.''08$ — $77^\circ 26' 30''$? [6: Part. IV, Vol. XI, p.386, 1860]

$2(\theta_4(5,1) - \theta_2(5,1)) = 135^\circ 27' 24.''48$ — $135^\circ 27' 55''$ [6: Vol. IV, p.108, 1870]

$2(\theta_3(5,1) - \theta_1(5,1)) = 141^\circ 10' 07.''96$ — $141^\circ 10' 05''$ [6: Part. IV, Vol. XI, p.277, 1866

$$\boxed{\Theta_{5,2}(\theta) = \sin^2\theta\cos\theta(\cos^2\theta - 1/3)}$$

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a) Zeros (the second and the fourth are equal to zeros of $\Theta_{2,0}(\theta)$):

$O_1(5,2) = 0^\circ$

$O_2(5,2) = \arccos\left(\frac{1}{\sqrt{3}}\right) = 54^\circ 44' 08.''20$ — Haüy: $54^\circ 44'$ [5: p.27]

$O_3(5,2) = 90^\circ$

$O_4(5,2) = \arccos\left(-\frac{1}{\sqrt{3}}\right) = 125^\circ 15' 51.''80$ — $125^\circ 15' 52''$ [6: Part I, Vol. III, p.364, 1853]

$O_5(5,2) = 180^\circ$

b) Sectors:

$O_3(5,2) - O_2(5,2) = 35^\circ 15' 51.''80$ — $35^\circ 23' 53''$? [6: Part IV, Vol. XII, p.631, 1860]

$35^\circ 07' 52''$? [6: Part IV, Vol. XII, p.631, 1860]

Average value $35^\circ 15' 52''$

$2(O_3(5,2) - O_2(5,2)) = O_4(5,2) - O_2(5,2) = 70^\circ 31' 43.''60$ — Haüy: $70^\circ 31' 44''$ [5: p.29; 1*]

$2O_2(5,2) = 109^\circ 28' 16.''40$ — Haüy: $109^\circ 28' 16''$ [5: p.29; 1*]

$O_5(5,2) - O_4(5,2) = 54^\circ 31' 43.''60$ — $54^\circ 32' 30''$ [6: Vol. V, p.304, 1870]

$2(O_4(5,2) - O_2(5,2)) = 141^\circ 03' 27.''20$ — $141^\circ 05' 55''$ [6: Vol. IV, p.102, 1870]

$141^\circ 00' 27''$ [6: Vol. IV, p.114, 1870]

Average value $141^\circ 3' 11''$

c) Extremes:

$$\theta_1(5,2) = \arccos\left(\sqrt{\frac{2}{5} + \sqrt{\left(\frac{2}{5}\right)^2 - \frac{1}{15}}}\right) = 32^\circ 51' 57.''05,$$

$32^\circ 46' 58''$? [6: Part IV, Vol. XII, p.629, 1860]

$$\theta_2(5,2) = \arccos\left(\sqrt{\frac{2}{5} - \sqrt{\left(\frac{2}{5}\right)^2 - \frac{1}{15}}}\right) = 72^\circ 05' 50.''53,$$

$72^\circ 06' 46''$ [6: Part II, Vol. IV, p.33, 1857]

$$\theta_3(5,2) = \arccos\left(-\sqrt{\frac{2}{5} - \sqrt{\left(\frac{2}{5}\right)^2 - \frac{1}{15}}}\right) = 107^\circ 54' 09.''47,$$ Lewis : $107^\circ 59' 30''$? [13]

$$\theta_4(5,2) = \arccos\left(-\sqrt{\frac{2}{5} + \sqrt{\left(\frac{2}{5}\right)^2 - \frac{1}{15}}}\right) = 147^\circ 08' 02.''95,$$

$147^\circ 08' 00''$ [6: Part I, Vol. II, p.298, 1870]

b) Sectors:

$2\theta_1(5,2) = 65^\circ 43' 54.''10$, — $65^\circ 43' 30''$ [6: Part IV, Vol. XI, p.251, 1866]

$65^\circ 44' 09''$ [6: Part IV, Vol. XI, p.251, 1866]

Average value $65^\circ 43' 49''$

$180^\circ - 2\theta_1(5,2) = \theta_4(5,2) - \theta_1(5,2) = 114^\circ 16' 05.''90$, $114^\circ 16' 00''$ [6: Part IV, Vol. XI, p.250, 1866]

$114^\circ 16' 15''$ [6: Part IV, Vol. X, p.160, 1860]

Average value $114^\circ 16' 8''$

$2\theta_2(5,2) = 144^\circ 11' 41.''06$ — $144^\circ 11'$ [6: Part. I, Vol. II, p.307, 1870]

$180^\circ - 2\theta_2(5,2) = \theta_3(5,2) - \theta_2(5,2) = 35^\circ 48' 18.''94$ — $35^\circ 45' 10''$? [6: Vol. IX, p.495, 1870]

$\theta_2(5,2) - \theta_1(5,2) = 42^\circ 13' 53.''48$ — $42^\circ 19' 28''$? [6: Part IV, Vol. XII, p.628, 1860]

$42^\circ 05' 51''$? [6: Part IV, Vol. XII, p.628, 1860]

Average value $42^\circ 12' 39''$

$180^\circ - (\theta_2(5,2) - \theta_1(5,2)) = 137^\circ 46' 6.''52$ — Glinka $137^\circ 45' 30''$ [10: p.52]

$\Delta\theta_6 = \theta_3(5,2) - \theta_1(5,2) = 75^\circ 02' 12.''42$ — Penfield: $75^\circ 02'$ [14]

$$\boxed{\Theta_{5,3}(\theta) = \sin^3\theta(\cos^2\theta - 1/9)}$$

---

a) Zeros:

$O_1(5,3) = 0^\circ$

$O_2(5,3) = \arccos\left(\frac{1}{3}\right) = 70^\circ 31' 43.''61$, — Haüy: $70^\circ 31' 44''$ [5: p.29; 1*]

$O_3(5,3) = \arccos\left(-\frac{1}{3}\right) = 109^\circ 28' 16.''39$ — Haüy: $109^\circ 28' 16''$ [5: p.29; 1*]

$O_4(5,3) = 180^\circ$

b) Sectors

$O_3(5,3) - O_2(5,3) = 38^\circ 56' 32.''78$ — $38^\circ 51' 32''$? [6: Part IV, Vol. XII, p.630, 1860]

$2(O_3(5,3)-O_2(5,3)) = 77^\circ\,53'\,5.''56$ — $77^\circ 53' 34''$ [6: Vol. IV, p.100, 1870]

$2O_2(5,3) = 360^\circ - 2O_3(5,3) = 141^\circ\,03'\,27.''22$, — $141^\circ 05' 55''$ [6: Vol. IV, p.102, 1870]

$141^\circ 00' 27''$ [6: Vol. IV, p.114, 1870]

Average value $141^\circ 3' 11''$

c) Extremes:

$\theta_1(5,3) = \arccos\left(\sqrt{\frac{7}{15}}\right) = 46^\circ 54'\,40.''60$ — $46^\circ 55'$ [6: Part III, Vol. VII, p.69, 1869]

$\theta_2(5,3) = 90^\circ$

$\theta_3(5,3) = \arccos\left(-\sqrt{\frac{7}{15}}\right) = 133^\circ 05' 19.''40$ — $133^\circ 09' 57''$? [6: Vol. IV, p.102, 1870]

$133^\circ 02'$? [6: Vol. IV, N.10, p.67, 1889]

Average value $133^\circ 05' 58''$

d) Sectors

$2\theta_1(5,3) = 93^\circ\,49'\,21.''20$ — $93^\circ 58' 0''$? [6: Part I, Vol. I, p.14, 1853]

$180^\circ - 2\theta_1(5,3) = 2(\theta_2(5,3)-\theta_1(5,3)) = \theta_2(5,3)-\theta_1(5,3) = 86^\circ\,10'\,38.''80$

Erofeev: $86^\circ 10' 38''$ [8: p.270]

$\theta_2(5,3)-\theta_1(5,3) = 43^\circ\,05'\,19.''40$ — $86^\circ 10' 38'' / 2 = 43^\circ 05' 19''$

$180^\circ - (\theta_3(5,3)-\theta_1(5,3)) = 103^\circ\,49'\,21.''20$ — $103^\circ 49' 12''$ [6: Part IV, Vol. X, p.138, 1860]

$180^\circ - (\theta_2(5,3)-\theta_1(5,3)) = 136^\circ\,54'\,40.''60$ — $136^\circ 52' 54''$ [6: Vol. I, N. 1, p.105, 1877]

$136^\circ 58' 20''$? [6: Part IV, Vol. X, p.101, 1860]

Average value $136^\circ 55' 37''$

$$\boxed{\Theta_{5,4}(\theta) = \sin^4\theta\cos\theta}$$

a) Zeros

$O_1(5,4) = 0^\circ$, $O_2(5,4) = 90^\circ$, $O_3(5,4) = 180^\circ$

b) Extremes

$\theta_1(5,4) = \theta_2(3,0) = O_2(3,1)$

$\theta_2(5,4) = \theta_3(3,0) = O_3(3,1)$

$$\boxed{\Theta_{6,0}(\theta) = \cos^6\theta - {}^{15}\!/_{11}\cos^4\theta + {}^{5}\!/_{11}\cos^2\theta - {}^{5}\!/_{231}}$$

a) Zeros

$O_1(6,0) = 21^\circ 10' 31''$ — $21^\circ 11' 21''$ [6: Part IV, Vol. XI, p.391, 1860]

$O_2(6,0) = 48^\circ 36' 28''$ — $48^\circ 31' 02''$? [6: Part IV, Vol. XI, p.285, 1866]

$O_3(6,0) = 76^\circ 11' 42''$ — $76^\circ 11' 24''$ [6: Part IV, Vol. X, p.138, 1860]

$O_4(6,0) = 103^\circ 48' 18''$ — $103^\circ 48' 36''$ [6: Part IV, Vol. XI, p.138, 1860]

$O_5(6,0) = 131^\circ 23' 32''$ — $131^\circ 24' 00''$ [6: Part IV, Vol. XI, p.252, 1866]

$O_6(6,0) = 158^\circ 49' 29''$ — $158^\circ 38' 40''$? [6: Part IV, Vol. X, p.112, 1860]

b) sectors

$2O_1(6,0) = 42^\circ 21' 02''$, — $42^\circ 19' 28''$ [6: Part IV, Vol. XII, p.630, 1860]

$180^\circ - 2O_1(6,0) = 137^\circ 38' 58''$ — $137^\circ 37' 45''$ [6: Part IV, Vol. X, p.107, 1860]

$180^\circ - 2O_2(6,0) = 82^\circ 47' 04''$ — Gordon: $82^\circ 48'$ [15]

$2O_2(6,0) = 97^\circ 12' 56''$, — $180^\circ - 82^\circ 48' = 97^\circ 12'$

$O_2(6,0) - O_1(6,0) = 27^\circ 25' 57''$, — $27^\circ 22' 30''$ [6: Part IV, Vol. X, p.159, 1860]

$180^\circ - (O_2(6,0) - O_1(6,0)) = 152^\circ 34' 3''$ — $152^\circ 34'$ [6: Part IV, Vol. X, p.161, 1860]

$152^\circ 34' 45''$ [6: Part IV, Vol. XII, p.626, 1860]

$O_3(6,0) - O_1(6,0) = 55^\circ 01' 11''$ — Goldschmidt, Peacock: $55^\circ 01' 30''$ [16]

$54^\circ 59' 52''$? [6: Part III, Vol. XII, p.626, 1860]

$180^\circ - (O_3(6,0) - O_1(6,0)) = 124^\circ 58' 49''$ — $124^\circ 57' 30''$ [6: Vol. IX, p.482,1870]

$O_4(6,0) - O_1(6,0) = 82^\circ 37' 47''$, — $82^\circ 37' 47''$ [6: Part I, Vol. III, p.431, 1870]

$180^\circ - (O_4(6,0) - O_1(6,0)) = 97^\circ 22' 13''$ — $180^\circ - 82^\circ 37' 47'' = 97^\circ 22' 13''$

$O_5(6,0) - O_1(6,0) = 110^\circ 13' 01''$, — $110^\circ 13' 14''$ [6: Part IV, Vol. XII, p.627, 1860]

$180^\circ - (O_5(6,0) - O_1(6,0)) = 69^\circ 46' 59''$ — $69^\circ 58' 47''$? [6: Part IV, Vol. XII, p.516, 1866]

$69^\circ 33' 14''$? [6: Part IV, Vol. XII, p.516, 1866]

Average value $69^\circ 46' 30''$

c) Extremes

$\theta_1(6,0) = 0^\circ$

$$\theta_2(6,0) = \arccos\left(\sqrt{\frac{5}{11} + \sqrt{\left(\frac{5}{11}\right)^2 - \frac{5}{33}}}\right) = 33^\circ\,52'\,41.''72$$

$33^\circ 56' 05''$ [6: Part IV, Vol. XII, p.516, 1866]

$33^\circ 49' 52''$ [6: Part IV, Vol. XII, p.516, 1866]

Average value $33^\circ 52' 58''$

$$\theta_3(6,0) = \arccos\left(\sqrt{\frac{5}{11} - \sqrt{\left(\frac{5}{11}\right)^2 - \frac{5}{33}}}\right) = 62^\circ 02' 25.''46$$

Goldschmidt, Palache, Peacock: $62^\circ 01'$ [17]

$62^\circ 03' 44''$ [6: Vol. III, p.100, 1870]

Average value $62^\circ 02' 22''$

$\theta_4(6,0) = 90^\circ$

$$\theta_5(6,0) = \arccos\left(-\sqrt{\frac{5}{11} - \sqrt{\left(\frac{5}{11}\right)^2 - \frac{5}{33}}}\right) = 117^\circ 57' 34.''54$$

Lebedev: $117^\circ 54'$ [11: p.270]

$$\theta_6(6,0) = \arccos\left(-\sqrt{\frac{5}{11}+\sqrt{\left(\frac{5}{11}\right)^2-\frac{5}{33}}}\right) = 146^\circ 07' 18.''28$$

$146^\circ 06' 28''$ [6: Part IV, Vol. XI, p.383, 1860]

$\theta_7(6,0) = 180^\circ$

d) Sectors:

$2\theta_2(6,0) = 67^\circ 45' 23.''44$ — $67^\circ 43' 20''$ [6: Vol. VIII, p.255, 1870]

$67^\circ 47' 30''$ [6: Vol. VIII, p.255, 1870]

Average value $67^\circ 45' 25''$

$180^\circ - 2\theta_2(6,0) = \theta_6(6,0) - \theta_2(6,0) = 112^\circ 14' 36.''56$ — $112^\circ 14' 30''$ [6: Vol. VIII, p.265, 1870]

$2\theta_3(6,0) = 124^\circ 04' 50.''92$ — $124^\circ 01' 45''$ [6: Vol. IV, p.107, 1870]

Palache: $124^\circ 07'$ [18]

Average value $124^\circ 04' 22''$

$180^\circ - 2\theta_3(6,0) = 55^\circ 55' 09.''68$ — $55^\circ 56' 03''$ [6: Part IV, Vol. XI, p.285, 1866]

$\theta_3(6,0) - \theta_2(6,0) = 28^\circ 09' 43.''74$ — Gordon: $28^\circ 07'$ [19]

Goldschmidt: $28^\circ 11' 30''$ [12: 7, p.139, 1922]

average value $28^\circ 09' 15''$

$180^\circ - (\theta_3(6,0) - \theta_2(6,0)) = 151^\circ 50' 16.''26$ — $151^\circ 50'$ [6: Part I, Vol. II, p.296, 1870]

$\theta_4(6,0) - \theta_2(6,0) = 56^\circ 07' 18.''28$ — $56^\circ 04' 57''$ [6: Vol. I, N.1, p.104, 1877]

$180^\circ - (\theta_4(6,0) - \theta_2(6,0)) = 123^\circ 52' 41.''72$ — $180^\circ - 56^\circ 04' 57'' = 123^\circ 55' 03''$

$\theta_4(6,0) - \theta_3(6,0) = 27^\circ 57' 34.''54$ — Goldschmidt, Shannon, Tocady, Garces: $27^\circ 55'$? [20]

$180^\circ - (\theta_4(6,0) - \theta_3(6,0)) = 152^\circ 02' 25.''46$ — $180^\circ - 27^\circ 55' = 152^\circ 05'$

Kokscharov-son: $151^\circ 54' 50''$? [6: Vol. IV, N.11, p.222, 1879]

$152^\circ 17' 01''$? [6: Vol. II, N.6, p.324,1878]

Average value $152^\circ 02' 56''$

$\theta_5(6,0) - \theta_2(6,0) = 84^\circ 04' 52.''82$ — Kokscharov-son: $83^\circ 54' 56''$? [6: Vol. IV, N.11. p. 223, 1879]

$180^\circ - (\theta_5(6,0) - \theta_2(6,0)) = 95^\circ 55' 7.''18$ — $95^\circ 54' 50''$ [6: Part IV, Vol. XI, p.258, 1866]

$\theta_5(6,0) - \theta_3(6,0) = 55^\circ 55' 09.''08$ — $55^\circ 56' 03''$ [6: Part IV, Vol. XI, p.285, 1866]

$180^\circ - (\theta_5(6,0) - \theta_3(6,0)) = 124^\circ 04' 50.''92$ — Palache: $124^\circ 07'$ [18]

$124^\circ 01' 45''$ [6: Vol. IV, p.107, 1870]

Average value $124^\circ 04' 22''$

$$\boxed{\Theta_{6,1}(\theta) = \sin\theta\cos\theta(\cos^4\theta - {}^{10}\!/_{11}\cos^2\theta + {}^{5}\!/_{33})}$$

---

a) Zeros are equal to extremes $\Theta_{6,0}(\theta)$

b) Extremes:

$\theta_1(6,1) = 16^\circ 22' 15.''47$ — $16^\circ 22' 58''$ [6: Part IV, Vol. XI, p.396, 1860]

$\theta_2(6,1) = 47^\circ 20' 49.''20$ — $47^\circ 17' 17''$ [6: Part IV, Vol. IV, p.101, 1860]

Average value $47^\circ 21' 33''$

$\theta_3(6,1) = 75^\circ 51' 03.''44$ — $75^\circ 51' 27''$ [6: Part IV, Vol. XI, p.390, 1860]

$\theta_4(6,1) = 104^\circ 08' 56.''56$ — $104^\circ 08' 57''$ [6: N. IV, p.102, 1870]

$\theta_5(6,1) = 132^\circ 39' 10.''80$ — $132^\circ 45' 10''$ [6: Vol. III, N.8, p.290, 1887]

$132^\circ 46' 40''$ [6: Vol. IV, N.10, p.52, 1889]

$132^\circ 21' 18''$ [6: Vol. III, N.8, p.296, 1887]

Average value $132^\circ 37' 42''$

$\theta_6(6,1) = 163^\circ 37' 44.''53$ — $163^\circ 37' 02$ [6: Part IV, Vol. XI, p.433, 1860]

c) Sectors:

$\theta_4(6,1) - \theta_3(6,1) = 28^\circ 17' 53.''12$ — $28^\circ 14' 04''$ [6: N.3, p.100, 1870]

$\theta_5(6,1) - \theta_2(6,1) = 85^\circ 18' 21.''60$ — $85^\circ 20' 16''$ [6: N.2, p.308, 1870]

$85^\circ 15' 27''$ [6: N.3, p.100, 1870]

Average value $85^\circ 17' 51''$

$\theta_6(6,1) - \theta_1(6,1) = 147^\circ 15' 29.''06$ — $147^\circ 08' 46''$ [6: Part IV, Vol. XI, p.434, 1860]

$147^\circ 23' 13''$ [6: Part I, Vol. I, p.18, 1853]

Average value $147^\circ 15' 59''$

$\theta_2(6,1) - \theta_1(6,1) = 30^\circ 58' 33.''73$ — $30^\circ 50' 35''$ [6: N.3, p.99, 1870]

$\theta_3(6,1) - \theta_1(6,1) = 59^\circ 28' 47.''97$ — $59^\circ 16' 46'$ [6: Part I, Vol. II, p.178, 1853]

$\theta_4(6,1) - \theta_1(6,1) = 87^\circ 46' 41.''09$ — $87^\circ 45' 20''$ [6: Part I, Vol. II, p.67, 1853]

$\theta_5(6,1) - \theta_1(6,1) = 116^\circ 16' 55.''33$ — $116^\circ 17'$ [6: Part I, Vol. II, p.76, 1853]

$$\boxed{\Theta_{6,2}(\theta) = \sin^2\theta(\cos^4\theta - {}^{6}\!/_{11}\cos^2\theta + {}^{1}\!/_{33})}$$

---

a) Zeros:

$O_1(6,2) = 0^\circ$

$$O_2(6,2) = \arccos\left(\sqrt{\frac{3}{11}+\sqrt{\left(\frac{3}{11}\right)^2-\frac{1}{33}}}\right) = 45^\circ 59' 34.''70$$

$45^\circ 57' 43''$ [6: Part IV, Vol. XI, p.287, 1866]

$46^\circ 01' 10''$ [6: Part IV, Vol. XI, p.287, 1866]

Average value $45^\circ 59' 27''$

$$O_3(6,2) = \arccos\left(\sqrt{\frac{3}{11}-\sqrt{\left(\frac{3}{11}\right)^2-\frac{1}{33}}}\right) = 75^\circ 29' 21.''05$$

$75^\circ 28' 15''$ [6: Part IV, Vol. X, p.160, 1860]

$75^\circ 32' 24''$ [6: Vol. II, N.6, p.321, 1878]

Average value $75^\circ 30' 20''$

$$O_4(6,2) = \arccos\left(-\sqrt{\frac{3}{11} - \sqrt{\left(\frac{3}{11}\right)^2 - \frac{1}{33}}}\right) = 104^\circ 30' 38.''95$$

$104^\circ 30'$ [6: Part IV, Vol. X, p.161, 1860]

$104^\circ 31' 40''$ [6: Part IV, Vol. X, p.159, 1860]

Average value $104^\circ 30' 50''$

$$O_5(6,2) = \arccos\left(-\sqrt{\frac{3}{11} + \sqrt{\left(\frac{3}{11}\right)^2 - \frac{1}{33}}}\right) = 134^\circ 00' 25.''30$$

$134^\circ 00' 30''$ [6: Part III, Vol. VII, p.60, 1853]

$O_6(6,2) = 180^\circ$

b) Sectors

$180^\circ - 2O_2(6,2) = 88^\circ 00' 50.''60$ — Lebedev: 88° [11: p.273]

$2O_2(6,2) = 91^\circ 59' 09.''40$ — $180^\circ - 88^\circ = 92^\circ$

$2O_3(6,2) = 150^\circ 58' 42.''10$, — $150^\circ 58'$ [6: Part IV, Vol. XI, p.404, 1860]

$180^\circ - 2O_3(6,0) = 29^\circ 01' 17.''90$ — $29^\circ 02' 40''$ [6: Part I, Vol. III, p.421, 1870]

$O_3(6,2) - O_2(6,2) = 29^\circ 29' 46.''35$, — $29^\circ 29' 16''$ [6: Part IV, Vol. XI, p.286, 1866]

$180^\circ - (O_3(6,2) - O_2(6,2)) = 150^\circ 30' 13.''65$ — $150^\circ 29' 45''$ [6: Part III, Vol. VII, p.43, 1853]

$O_4(6,2) - O_2(6,2) = 58^\circ 31' 04.''25$, — Haüy: $58^\circ 31' 04''$ [5: p.85; 2*]

$180^\circ - (O_4(6,2) - O_2(6,2)) = 121^\circ 28' 55.''75$, — Haüy: $121^\circ 28' 56''$ [5: p.86; 2*]

c) Extremes:

$$\theta_1(6,2) = \arccos\left(\sqrt{\frac{17}{33} + \sqrt{\left(\frac{17}{33}\right)^2 - \frac{19}{99}}}\right) = 27^\circ 32' 30.''54$$

$27^\circ 39' 38''$? [6: Part IV, Vol. X, p.99, 1860]

$$\theta_1(6,2) = \arccos\left(\sqrt{\frac{17}{33} - \sqrt{\left(\frac{17}{33}\right)^2 - \frac{19}{99}}}\right) = 60^\circ 23' 27.''73$$

$60^\circ 24' 10''$ [6: Part IV, Vol. XI, p.261, 1866]

$\theta_3(6,2) = 90^\circ$

$$\theta_4(6,2) = \arccos\left(-\sqrt{\frac{17}{33} - \sqrt{\left(\frac{17}{33}\right)^2 - \frac{19}{99}}}\right) = 119^\circ 36' 32.''27$$

Glinka: $119^\circ 36'$ [10: p.67]

$$\theta_5(6,2) = \arccos\left(-\sqrt{\frac{17}{33} + \sqrt{\left(\frac{17}{33}\right)^2 - \frac{19}{99}}}\right) = 152^\circ 27' 29.''46$$

$152^\circ 28' 34''$ [6: Part IV, Vol. X, p.99, 1860]

d) Sectors

$2\theta_1(6,2) = 55^\circ 05' 01.''08$ — Fletcher: $55^\circ 06'$ [21]

$180^\circ - 2\theta_1(6,2) = \theta_5(6,2) - \theta_1(6,2) = 124^\circ 58' 58.''92$

$124^\circ 58' 50''$ [6: Part IV, Vol. X, p.100, 1860]

$2\theta_2(6,2) = 120^\circ 46' 55.''46$ — Glinka: $120^\circ 46'$ [10: p.65]

$180^\circ - 2\theta_2(6,2) = 59^\circ 13' 04.''54$ — $180^\circ - 120^\circ 46' = 59^\circ 14'$

$\theta_5(6,2) - \theta_1(6,2) = 125^\circ 54' 58.''92$ — $125^\circ 57'$ [6: Part IV, Vol. XI, p.624, 1860]

$180^\circ - (\theta_5(6,2) - \theta_1(6,2)) = 54^\circ 5' 01.''08$ — $53^\circ 59' 37''$? [6: Part IV, Vol. XI, p.286, 1866]

$\theta_3(6,2) - \theta_1(6,2) = 62^\circ 27' 29.''46$ — $62^\circ 35' 16''$? [6: Part IV, Vol. XI, p.286, 1866]

$62^\circ 12' 58''$? [6: Part IV, Vol. XI, p.286, 1866]

Average value $62^\circ 24' 7''$

$180^\circ - (\theta_3(6,2) - \theta_1(6,2)) = 117^\circ 32' 30.''54$ — $117^\circ 34'$? [6: Part IV, Vol. X, p.100, 1860]

$\theta_4(6,2) - \theta_1(6,2) = 92^\circ 4' 1.''73$ — $92^\circ 9' 30''$? [6: Part IV, Vol. X, p.87, 1860]

$180^\circ - (\theta_4(6,2) - \theta_1(6,2)) = 87^\circ 55' 58.''27$ — $87^\circ 50' 30''$? [6: Part IV, Vol. X, p.87, 1860]

$\theta_2(6,2) - \theta_1(6,2) = 32^\circ 50' 57.''19$ — $32^\circ 51' 14''$ [6: Part IV, Vol. XI, p.396, 1860]

$180^\circ - (\theta_2(6,2) - \theta_1(6,2)) = 147^\circ 9' 2.''81$ — $147^\circ 9'$ [6: Part IV, Vol. X, p.161, 1860]

$\theta_3(6,2) - \theta_2(6,2) = 19^\circ 36' 32.''27$ — $180^\circ - 160^\circ 22' 35''$

$180^\circ - (\theta_3(6,2) - \theta_2(6,2)) = 160^\circ 23' 27.''73$ — $160^\circ 22' 35''$ [6: Vol. IV, p.106, 1870]

$\theta_4(6,2) - \theta_2(6,2) = 59^\circ 13' 04.''54$ — $59^\circ 23' 30''$? [6: Vol. VII, p.254, 1870]

$180^\circ - (\theta_4(6,2) - \theta_2(6,2)) = 120^\circ 46' 55.''46$ — $120^\circ 44'$? [6: Vol. IX, p.486, 1870]

$$\boxed{\Theta_{6,3}(\theta) = \sin^3\theta\cos\theta(\cos^2\theta - 3/11)}$$

a) Zero:

$O_1(6,3) = 0^\circ$

$O_2(6,3) = \arccos\sqrt{\frac{3}{11}} = 58^\circ 31' 04.''25$ — Haüy: $58^\circ 31' 04''$ [5: p.85; 2*]

$O_3(6,3) = 90^\circ$

$O_4(6,3) = \arccos\left(-\sqrt{\frac{3}{11}}\right) = 121^\circ 28' 55.''75$ — Haüy: $121^\circ 28' 56''$ [5: p.86; 6*]

$O_5(6,3) = 180^\circ$

b) Sectors:

$\Delta\theta_1 = 2O_2(6,3) = 117^\circ 02' 08''.50$, — Haüy: $117^\circ 02' 08''$ [5: p.85; 2*]

$180^\circ - 2O_1(6,3) = 62^\circ 57' 51.''1$ — Eakle: $62^\circ 59'$ [22]

$O_4(6,3) - O_2(6,3) = 62^\circ 37' 51.''50$ — $62^\circ 38' 36''$ [6: Part III, Vol. VII, p.76, 1869]

$O_3(6,3)-O_2(6,3)=31^\circ 28'55.''75$ — $31^\circ 30'54''$ [6: Vol. IV, p.100, 1870]

$180^\circ-(O_3(6,3)-O_2(6,3))=148^\circ 31'04.''25$ — $148^\circ 30'$ [6: Part I, Vol. III, p.341, 1853]

c) Extremes:

$$\theta_1(6,3)=\arccos\left(\sqrt{\frac{5}{11}+\sqrt{\left(\frac{5}{11}\right)^2-\frac{1}{22}}}\right)=22^\circ 18'07.''02$$

$22^\circ 20'55''$ [6: Part IV, Vol. XI, p.389, 1860]

$$\theta_2(6,3)=\arccos\left(\sqrt{\frac{5}{11}-\sqrt{\left(\frac{5}{11}\right)^2-\frac{1}{22}}}\right)=76^\circ 40'37.''65$$

$76^\circ 42'39''$ [6: Part I, Vol. III, p.433, 1870]

$$\theta_3(6,3)=\arccos\left(-\sqrt{\frac{5}{11}-\sqrt{\left(\frac{5}{11}\right)^2-\frac{1}{22}}}\right)=103^\circ 19'22.''35$$

$103^\circ 20'40''$ [6: Part IV, Vol. XI, p.418, 1860]

$$\theta_4(6,3)=\arccos\left(-\sqrt{\frac{5}{11}+\sqrt{\left(\frac{5}{11}\right)^2-\frac{1}{22}}}\right)=157^\circ 41'52.''98$$

$157^\circ 42'43''$ [6: Vol. IV, p.111, 1870]

$157^\circ 40'59''$ [6: Vol. I, N.1, p.113, 1877]

Average value $157^\circ 41'51''$

d)Sectors:

$2\theta_1(6,3)=44^\circ 36'14.''04$ — $44^\circ 36'20''$ [6: Vol. V, p.304, 1870]

$180^\circ-2\theta_1(6,3)=\theta_4(6,3)-\theta_1(6,3)=135^\circ 23'45.''96$ — $135^\circ 18'46''$? [6: Vol. IV, p.102, 1870]

$135^\circ 29'30''$? [6: Part I, Vol. III, p.346, 1853]

Average value $135^\circ 24'8''$

$2\theta_2(6,3)=153^\circ 21'15.''30$ — $153^\circ 19'30''$ [6: Part I, Vol. III, p.428, 1870]

$153^\circ 26'6''$? [6: Part I, Vol. III, p.334, 1853]

$153^\circ 17'31''$? [6: Part IV, Vol. X, p.144, 1860]

average value of the last two angles $153^\circ 21'48''$

$180^\circ-2\theta_2(6,3)=26^\circ 38'44.''70$ — $26^\circ 38'$ [6: Part I, Vol. III, p.428, 1870]

$\theta_2(6,3)-\theta_1(6,3)=54^\circ 22'30.''63$ — $54^\circ 22'04''$ [6: Part III, Vol. VII, p.99, 1869]

$180^\circ-(\theta_2(6,3)-\theta_1(6,3))=125^\circ 37'29.''37$ — $125^\circ 31'32''$? [6: Part IV, Vol. XI, p.387, 1860]

$\theta_3(6,3)-\theta_1(6,3)=81^\circ 01'15.''33$ — $81^\circ 03'07''$ [6: Part IV, Vol. XI, p.389, 1860]

$180^\circ-(\theta_3(6,3)-\theta_1(6,3))=98^\circ 58'44.''67$ — $98^\circ 56'53''$ [6: Part IV, Vol. XI, p.384, 1860]

$\boxed{\Theta_{6,4}(\theta)=\sin^4\theta(\cos^2\theta-1/11)}$

a) Zeros:

$O_1(6,4)=0^\circ$

$O_2(6,4)=\arccos\left(\frac{1}{\sqrt{11}}\right)=72^\circ 27'05.''76$ — $72^\circ 30'28''$ [6: Part I, Vol. III, p.349, 1853]

$O_3(6,4)=\arccos\left(-\frac{1}{\sqrt{11}}\right)=107^\circ 32'54.''24$ — $107^\circ 33'13''$ [6: Part IV, Vol. XI, p.281, 1866]

$O_4(6,4)=180^\circ$

b) Sectors:

$2O_2(6,4)=144^\circ 54'11.''52$ — $144^\circ 50'31''$? [6: Vol. IV, p.103, 1870]

$180^\circ-2O_2(6,4)=35^\circ 05'48.''48$ — $35^\circ 06'56''$ [6: Part IV, Vol. XI, p.389, 1860]

c) Extremes

$\theta_1(6,4)=\arccos\left(\sqrt{\frac{13}{33}}\right)=51^\circ 07'24.''04$ — $51^\circ 08'28''$ [5: Part IV, Vol. XII, p.630, 1860]

$\theta_2(6,3)=90^\circ$

$\theta_3(6,4)=\arccos\left(-\sqrt{\frac{13}{33}}\right)=128^\circ 52'35.''96$ — $128^\circ 54'02''$ [6: Vol. IV, p.104, 1870]

$128^\circ 50'50''$ [6: Vol. VIII, p.251, 1870]

Average value $128^\circ 52'35''$

d) Sectors:

$\Delta\theta_1=2\theta_1(6,4)=102^\circ 14'48.''08$ — $102^\circ 12'40''$ [6: Vol. IV, p.107, 1870]

$102^\circ 16'42''$? [9: 21]

Average value $102^\circ 14'41''$

$180^\circ-2\theta_1(6,4)=\theta_3(6,4)-\theta_1(6,4)=77^\circ 45'11.''92$ — Palache: $77^\circ 44'$? [23]

$\theta_2(6,4)-\theta_1(6,4)=38^\circ 52'35.''96$ — $38^\circ 51'32''$ [6: Part IV, Vol. XII, p.630, 1860]

$180^\circ-(\theta_2(6,4)-\theta_1(6,4))=151^\circ 07'24.''04$ — $151^\circ 03'16''$? [6: Part IV, Vol. XI, p.433, 1860]

$\boxed{\Theta_{6,5}(\theta)=\sin^5\theta\cos\theta}$

a) Zeros:

$O_1(6,5)=0^\circ$, $O_2(6,5)=90^\circ$, $O_4(6,5)=180^\circ$ — typical angles of crystals

c) Extremes

$\theta_1(6,5)=\arccos\left(\frac{1}{\sqrt{6}}\right)=65^\circ 54'18.''67$ — $65^\circ 51'28''$ [6: Part IV, Vol. XI, p.393, 1860]

$65^\circ 58'13''$ [6: Part IV, Vol. XI, p.388, 1860]

Average value $65^\circ 54' 50''$

$\theta_2(6,5) = \arccos\left(-\frac{1}{\sqrt{6}}\right) = 114^\circ 05' 41.''33$ $\qquad$ $114^\circ 01' 47''$ [6: Part IV, Vol. XI, p.431, 1860]

d) Sectors:

$\Delta\theta_1 = 2\theta_1(6,4) = 131^\circ 48' 37.''34$ $\qquad$ Kokscharov-son: $131^\circ 57' 28''$? [6: Vol. IV, N.11, p.223, 1879]

$\Delta\theta_1 = \pi - 2\theta_1(6,4) = 48^\circ 11' 22.''66$ $\qquad$ $48^\circ 07' 41''$? [6: Part IV, Vol. XI, p.393, 1860]

$48^\circ 16' 16''$? [6: Part IV, Vol. XI, p.389, 1860]

Average value $48^\circ 11' 58''$

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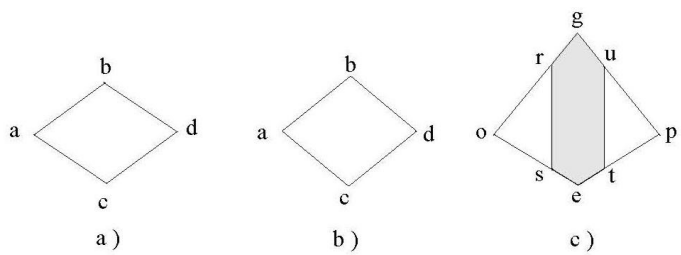
4. Facet angles of some crystals: examples

Let us consider finally the geometry of some crystals (Fig. 1) from the point of view of typical angles of polar functions resting upon Haüy's works [5].



$\angle rgu = 78^\circ 27' 47''$ → $2O_1(3,0) = 78^\circ 27' 46.''94$,

$\angle set = 117^\circ 02' 8''$ → $2O_2(6,3) = 117^\circ 02' 8.''50$,

$\angle gut = \angle grs = 140^\circ 46' 07''$ → $O_3(3,0) = 140^\circ 46' 06.''53$

$\angle rse = \angle etu = 121^\circ 28' 56''$ → $O_4(6,3) = 121^\circ 28' 55.''75$

$\angle goe = \angle gpe = 82^\circ 15' 03''$ → $180^\circ - (O_1(3,0) + O_2(6,3)) = 82^\circ 15' 02.''28$

Fig. 1. Rhombic facets of some crystals.

a) The pomegranate with 24 facets [5: p.82; 2*]. The scanning is 24 rhombuses. The angles of the rhombuses (Fig. 1a) are correspondingly equal to [5: p.79; 1*]:

$\angle bac = 70^\circ 31' 44''$ → $O_2(5,3) = O_2(2,0) - O_1(2,0) = 70^\circ 31' 43.''60$,

$\angle acd = 109^\circ 28' 16''$ → $O_3(5,3) = 2O_1(2,0) = 109^\circ 28' 16.''40$

b) The lime spar [5: 36, 7*]. The scanning is 6 rhombuses (Fig. 1b) with angles:

$\angle bac = 78^\circ 27' 47''$ → $2O_1(3,0) = 78^\circ 27' 46.''94$,

$\angle acd = 101^\circ 32' 13''$ → $O_3(3,0) - O_1(3,0) = 2(O_3(3,0) - O_2(3,0)) = 101^\circ 32' 13.''06$

c) The pomegranate with 36 facets. The scanning is 12 rhombuses (Fig. 1a) and 24 prolate hexagons ("*argute*", Fig. 1c) [5: 82; 2*]. The angles are:

Thus, the WM, developed by us to replace the Standard Model,

solved the problem

(indicated in the title of this report),

uncovered the above-mentioned riddle of crystals:

The nature of crystal formations, in fact, is wave.

This statement is based on the discovery of a previously unknown fact that

characteristic angles of the crystals,

determining their shape, completely

coincide with the

characteristic angles of the solutions for the

polar component

of the general (“classical”) **wave equation.**

This discovery once again confirms

the validity and advantage of the concepts underlying the Wave Model.

Conclusion

We highlight here the following

breakthrough discovery

(we made within the WM),

which alters conventional ideas regarding the structure of matter:

- **1. Discovery of the shell-nodal structure of the atoms.**

"Atoms" ($Z \geq 2$) are stable wave formations. They have the shell-nodal structure and are elementary molecules of hydrogen atoms ($Z=1$). Their nodes, filled with paired hydrogen atoms, are bound to each other by strong wave interaction (dependent on the fundamental frequency ω_e of atomic and subatomic levels [1]).

This discovery, as a fundamental key one,

led to a series of

derivative discoveries:

- **2. Discovery of the nature of chemical bonds of "atoms" - elementary molecules of H-atoms.**

The main role at the formation of specific configuration of molecules and crystals plays a spatial arrangement of the nodes and internodal strong bonds in the elementary molecules ("atoms") of hydrogen atoms, but not the so-called "electron configuration". Electrons play the secondary role: they define only the strength of chemical bonds.

Chemical "covalent" bonds are realized directly along strong internodal bonds each of the joined elementary molecules ("atoms") or their dimers.

- **3. Discovery of the original nature of "Atomic" Isotopes and their entire set (including those that have not yet been detected).**

It was made as a result of an analysis of particular solutions of the wave equation.

All "atomic" isotopes (natural and artificial, already detected and not yet detected), their structure and relative mass are defined by the extent of filling all potential and potential-kinetic nodes of the "atoms" with hydrogen atoms.

- **4. Discovery of the original nature of the Periodic Law.**

- **5. Discovery of the Periodic Table,** theoretical, consistent with particular solutions of the wave equation.

The original cause of the observed periodicity in properties of chemical elements is the quasi-periodicity of the shell-nodal structure of the "atoms", following from the wave equation solutions. With allowance of the above-mentioned found regularity, the "atoms" were arranged and, for the first time in physics, presented in the form of the theoretical Periodic Table of "Atoms", reflecting, thus, the primary cause of the periodicity.

Remind. The periodic table, or periodic table of elements, presented currently in physics, is a tabular arrangement of the chemical elements, arranged by atomic number, electron configuration, and recurring chemical properties, whose structure shows periodic trends.

- **6. Discovery of the wave nature of crystals.**

Characteristic angles of crystals of natural minerals are determined by the same particular solutions of the wave equation (2) just like the spatial angles of the disposition of nodes in the wave shells of the atoms.

Derivative discoveries

(originating from the breakthrough one)

related to graphene, fullerenes and nanotubes:

1. Discovery of anisotropy of two-dimensional crystal lattice of graphene.

Graphene has two-fold rotational symmetry, but not six-fold as is commonly believed.

2. Discovery of the nature of the "ballistic" conductivity in graphene.

The "ballistic" motion of charges in graphene is realized along parallel hollow channels in the graphene lattice, formed from invisible empty polar nodes.

3. Discovery of the direction of semiconductor conductivity in graphene.

The semiconductor feature takes place in the crystallographic direction perpendicular to the "ballistic" channels oriented along the Z-axis.

4. Discovery of the nature of the formation of bonds in the molecule of buckminsterfullerene and its true chemical formula $(C_2)_{30}$.

Elementary building blocks of the molecule are carbon dimers C_2 .

5. Discovery of the cause of the formation of carbon nanotubes with semiconductor and metallic conductivity.

Thus,

atoms and their compounds

are

the material realization

of elementary solutions of the wave equation.

About the wave structure and behaviour of elementary particles, can be found in the Selected Lectures on the WM by the author [13].

Remind

The Wave Model is based on the axiom

(consistent with dialectical philosophy), according to which

all material formations at all levels of the Universe, being harmonically interrelated between themselves, as everything in the Universe, have the wave nature.

Taking into account the totality of discoveries of the WM, made thanks to adequate concepts on which it is based,

we came to the following

Overall conclusion

Accepting the above axiom

as the basic concept for all theories of physics

can, ultimately, save modern physics from the subjective approach

(characterized by the use of abstract-mathematical, fictional, postulates)

and bring it

to a qualitatively new level in its development

This is a universal concept that all physicists were looking for !

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George P. Shpenkov

g.shpenkov@gmail.com

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Thank you
for your attention!